ORMC AMC 10/12 Group
Week 1: Algebra

October 1, 2023

1 Warm-up Exercises

1. Find all x such that $2x^2 + 5 = x^2 + 18$

   Solution: Isolate x^2 to get $x^2 = 13 \implies x = \pm\sqrt{13}$

2. Solve the following system of equations:

$$x - y = 3, \qquad \frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$$

   Solution: We can use the first equation to write $x = y + 3$, and then substitute that into the second equation to get:

$$\frac{1}{y+3} + \frac{1}{y} = \frac{1}{2} \implies y + y + 3 = \frac{1}{2}\left(y^2 + 3y\right)$$
$$\implies 4y + 6 = y^2 + 3y \implies y^2 - y - 6 = 0 \implies (y-3)(y+2) = 0 \implies y = 3 \text{ or } y = -2$$

   We can check that $y = 3, x = 6$ and $y = -2, x = 1$ both work as solutions.

3. Compute $x + y + z$:

$$x = y + z + 2,$$
$$y = z + x + 1,$$
$$z = x + y + 4.$$

   Solution: Let $A = x + y + z$. If we add all the equations together, we get:

$$A = 2A + 7 \implies x + y + z = A = -7$$

4. Find all a such that:

$$\frac{3}{1 - \sqrt{a-2}} + \frac{3}{1 + \sqrt{a-2}} = 6.$$

   Solution: Multiply through by $(1 - \sqrt{a-2})(1 + \sqrt{a-2}) = 1 - (a - 2) = 3 - a$ to remove the denominators. We get:

$$3 - 3\sqrt{a-2} + 3 + 3\sqrt{a-2} = 18 - 3a \implies 6 = 18 - 3a \implies 3a = 12 \implies a = 4.$$

5. Solve for x:

$$\begin{aligned} x + 2y - z &= 5, \\ 3x + 2y + z &= 11, \\ (x+2y)^2 - z^2 &= 15. \end{aligned}$$

Solution: Subtracting the first equation from the second eliminates y, and adding the two equations together eliminates z. Doing both of these, we get:

$$\begin{aligned} 2x + 2z = 6, &\qquad 4x + 4y = 16 \\ \implies z = 3 - x, &\qquad y = 4 - x. \end{aligned}$$

This allows us to substitute for y and z in the third equation, where we can solve for x. We get:

$$\begin{gathered} (x + 2(4-x))^2 - (3-x)^2 = 15 \implies (8-x)^2 - (3-x)^2 = 15 \\ \implies (64 - 16x + x^2) - (9 - 6x + x^2) = 15 \implies 55 - 10x = 15 \\ \implies 10x = 40 \implies x = 4. \end{gathered}$$

2 Algebraic Manipulations

Useful factoring identities:

- $(a+b)^2 = a^2 + 2ab + b^2$
  - $(x+1)^2 =$ ________________
    Solution: $x^2 + 2x + 1$.

- $a^2 - b^2 = (a+b)(a-b)$
  - Compute $63 \cdot 65$
    Solution: $63 \cdot 65 = (64-1)(64+1) = 64^2 - 1 = (2^6)^2 - 1 = 2^{12} - 1 = 4096 - 1 = 4095$.
  - What is the difference between 2 consecutive perfect squares, i.e. n^2 and $(n+1)^2$?
    Solution: using the difference of squares formula, we can see that

    $$(n+1)^2 - n^2 = ((n+1) - n)((n+1) + n) = ((n+1) + n) = 2n + 1,$$

    the sum of the two numbers that were squared.

- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
  - Given $a - b = 9$ and $ab = 7$, compute $a^3 - b^3$
    Solution: We just have to create the second term, which we can do by squaring $a - b$ and adding ab an appropriate amount of times. We have $9^2 = a^2 - 2ab + b^2$, so we add ab 3 times to get $a^2 + ab + b^2 = 81 + 21 = 102$. Then, $a^3 - b^3 = (a-b)(a^2 + ab + b^2) = 9 \cdot 102 = 918$.

- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  - Notice that $a^3 + b^3 = a^3 - (-b)^3 = (a - (-b))(a^2 + a(-b) + (-b)^2) = (a+b)(a^2 - ab + b^2)$

- $(a+1)(b+1) = ab + a + b + 1$
  - Find all pairs of integers x, y such that $xy + x + y = 9$
    Solution: The idea here is to "complete" the factorization by adding 1 to both sides. If we do that, then factor, we get $(x+1)(y+1) = 10$. Since x and y are both integers, the left hand side has to be some factorization of 10. The factor pairs of 10 are $(1, 10), (2, 5), (5, 2), (10, 1)$. So, this means that our solutions can be: $(0, 9), (1, 4), (4, 1), (9, 0)$.

  - Find all pairs of integers x, y such that $xy + 3x + 4y = 13$
    Solution: We follow the same procedure as the previous problem, but since the coefficients are 3 and 4, we add 12 instead of adding 1. So, we get $(x+4)(y+3) = 25$. As before, we start with the factor pairs, and subtract appropriately to get the proper values of x and y. The factor pairs are $(1, 25), (5, 5), (25, 1)$ so our possible solutions are $(-3, 22), (1, 2), (21, -2)$.

2.1 Examples

1. Given $3x + \frac{1}{2x} = 3$, find $8x^3 + \frac{1}{27x^3}$.

   Solution: The key is to first find $2x + \frac{1}{3x}$ by multiplying the entire first equation by $\frac{2}{3}$. This gives us $2x + \frac{1}{3x} = 2$. The desired expression is given by the following factorization:

$$8x^3 + \frac{1}{27x^3} = \left(2x + \frac{1}{3x}\right)\left((2x)^2 - 2x\frac{1}{3x} + \left(\frac{1}{3x}\right)^2\right)$$
$$= 2\left(\left(2x + \frac{1}{3x}\right)^2 - 3\left(2x\frac{1}{3x}\right)\right) = 2\left(4 - 3\left(\frac{2}{3}\right)\right) = 2(4-2) = 4.$$

2. Find all pairs of positive integers a, b which satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$.

   The denominators make this very hard to work with, so we can start by getting rid of all of them. Multiply the entire equation by $6ab$, and then use "Simon's favorite factoring trick":

$$6b + 6a = ab \implies ab - 6a - 6b = 0$$
$$\implies ab - 6a - 6b + 36 = 36 \implies (a-6)(b-6) = 36$$

   So, we just need to find factorizations of 36. The (unordered) factor pairs are $(1, 36), (2, 18), (3, 12), (4, 9), (6, 6)$. So, the corresponding (unordered) solution pairs are:

$$(7, 42), (8, 24), (9, 18), (10, 15), (12, 12).$$

2.2 Exercises

1. Find a factorization of $x^5 - y^5$, based on the given factorization for $x^3 - y^3$. Can you find a similar factorization for $x^5 + y^5$? Do the same factorizations work for $x^4 - y^4$ and $x^4 + y^4$?

   Solution: Notice that for $x^5 \pm y^5$ and $x^4 - y^4$, the same factorization methods work:

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$
$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$
$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3)$$

   However, we cannot do the same factorization for $x^4 + y^4$. The issue here is that,as noted above, the factorizations for $x^3 + y^3$ and $x^5 + y^5$ come from $x^3 - (-y)^3$, but when we substitute $(-y)$ into a 4^{th} power, the negative cancels itself out. So, the $x^n - y^n$ factorization method works for all n, but the $x^n + y^n$ factorization works only for odd n.

2. What is the product of all integers x for which $|x^2 - 9|$ is prime?

   Solution: Notice that $x^2 - 9$ factors as $(x - 3)(x + 3)$, and we can break up the absolute value to look like $|x - 3| \cdot |x + 3|$. So, either $x - 3$ or $x + 3$ must be ± 1, and the other one must be prime. If $x - 3 = \pm 1$, then we have $x = 2$ or 4, and we have $x = -2$ or -4 for the other case. If $x = \pm 2$ then we get $|x^2 - 9| = 5$, and if $x = \pm 4$ then we get $|x^2 - 9| = 7$, and both cases are prime. So the possible values are $\pm 2, \pm 4$, which gives us a total product of 64.

3. Compute

$$52683 \cdot 52683 - 52660 \cdot 52706.$$

   Solution: The idea is to convert the second product into a difference of two squares in such a way that it cancels out with the first product, which is a square. Notice that $52660 = 52683 - 23$ and $52706 = 52683 + 23$. So, the expression simplifies to $52683^2 - (52683^2 - 23^2) = 23^2 = 529$.

4. **(2008 AMC 10A #7)** Simplify

$$\frac{(3^{2008})^2 - (3^{2006})^2}{(3^{2007})^2 - (3^{2005})^2}.$$

   Solution: The easiest way to do this is to notice that we can factor two 3's out of the numerator to get the same thing as the denominator, which means that the total value of the expression is 9. If you aren't yet clear on how the layered exponents work, you can still use a difference-of-squares approach to get the answer:

$$\frac{(3^{2008})^2 - (3^{2006})^2}{(3^{2007})^2 - (3^{2005})^2} = \frac{(3^{2008} - 3^{2006})(3^{2008} + 3^{2006})}{(3^{2007} - 3^{2005})(3^{2007} + 3^{2005})}$$
$$= \frac{3(3^{2007} - 3^{2005}) \cdot 3(3^{2007} + 3^{2005})}{(3^{2007} - 3^{2005})(3^{2007} + 3^{2005})} = 3 \cdot 3 = 9.$$

5. **(2021 AMC 12A #9)** Which of the following is equivalent to

$$(2 + 3)(2^2 + 3^2)(2^4 + 3^4)(2^8 + 3^8)(2^{16} + 3^{16})(2^{32} + 3^{32})(2^{64} + 3^{64})?$$

   (A) $3^{127} + 2^{127}$ **(B)** $3^{127} + 2^{127} + 2 \cdot 3^{63} + 3 \cdot 2^{63}$ **(C)** $3^{128} - 2^{128}$ **(D)** $3^{128} + 2^{128}$ **(E)** 5^{127}

Solution: Notice that these are all "sums of squares". In particular, if we multiply through by (2-3), we get a cascading effect where the differences continue to combine into larger and larger differences of squares:

$$\begin{aligned}
&(2-3)(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\
&= (2^2-3^2)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\
&= (2^4-3^4)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\
&= (2^8-3^8)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\
&= (2^{16}-3^{16})(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\
&= (2^{32}-3^{32})(2^{32}+3^{32})(2^{64}+3^{64}) \\
&= (2^{64}-3^{64})(2^{64}+3^{64}) \\
&= (2^{128}-3^{128})
\end{aligned}$$

Since $2-3=-1$, the correct answer is the negative of this, **(C)** $3^{128}-2^{128}$.

The other thing that you may have noticed if you tried to start multiplying the terms out, is that once you've multiplied up to term $3^{2^x}+2^{2^x}$, the result looks like the longer term in the factorization of $3^{2^{x+1}}-2^{2^{x+1}}$. For example, 3^4-2^4 factors as

$$(3-2)(3^3+3^2\cdot 2+3\cdot 2^2+2^3),$$

and the second term is precisely the product $(3+2)(3^2+2^2)$. So, it would stand to reason that once we multiply them all out, we have the longer term from $3^{128}-2^{128}$, and the shorter term is simply $(3-2)=1$, so $3^{128}-2^{128}$ is in fact our final answer.

6. **(2008 AMC 12B #16)** A rectangular floor measures a by b feet, where a and b are positive integers with $b>a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. How many possibilities are there for the ordered pair (a,b)?

   Solution: The inner area without the border is $(a-2)(b-2)$, since we remove 1 foot on each side. The full area is ab. We are given that the border is half of the total area, so the inner non-border area is also half of the total area. So $ab=2(a-2)(b-2)=2ab-4a-4b+8$. Moving things around to use the factorization trick, we get:

   $$ab-4a-4b+8=0 \implies ab-4a-4b+16=8 \implies (a-4)(b-4)=8$$

   Since we must have $a<b$, we only need the number of unordered factor pairs of 8, which is

   $$(1,8),(2,4) \implies \boxed{2}.$$

7. Evaluate $2022^3 - 2022^2 \cdot 2023 - 2022 \cdot 2023^2 + 2023^3$

Solution: The tricky part here is to not get caught up with how the expression looks similar to the longer term in the difference-of-powers factorizations. If we pair the 1st and 3rd terms, and the 2nd and 4th terms, we can simply use difference of squares:

$$\begin{aligned}2022^3 - 2022^2 \cdot 2023 - 2022 \cdot 2023^2 + 2023^3 &= (2022^3 - 2022 \cdot 2023^2) - (2022^2 \cdot 2023 - 2023^3) \\ &= 2022(2022^2 - 2023^2) - 2023(2022^2 - 2023)^2 = (2022 - 2023)(2022^2 - 2023^2) \\ &= (2022 - 2023)^2(2022 + 2023) = 2022 + 2023 = \boxed{4045}.\end{aligned}$$

8. **(1987 AIME #5)** Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$

Solution: We can start by setting up for "Simon's favorite factoring trick":

$$3x^2y^2 + y^2 - 30x^2 = 517 \implies 3x^2y^2 + y^2 - 30x^2 - 10 = 507 \implies (3x^2 + 1)(y^2 - 10) = 507.$$

Since both x and y are integers, both of the factors on the left-hand side must also be integers, and 507 factors as $3 \cdot 13^2$. It is easy to see that 3 cannot divide $3x^2 + 1$, and that 169 does not work for $3x^2 + 1$, which means that $3x^2 + 1 = 13$ and $y^2 - 10 = 39$. So, $3x^2 = 12, y^2 = 49$ which gives us $12 \cdot 49 = \boxed{588}$ as our final answer.

9. **(2022 AMC 12A #21)** Let

$$P(x) = x^{2022} + x^{1011} + 1.$$

Which of the following polynomials is a factor of $P(x)$?

(A) $x^2 - x + 1$ **(B)** $x^2 + x + 1$ **(C)** $x^4 + 1$ **(D)** $x^6 - x^3 + 1$ **(E)** $x^6 + x^3 + 1$

Solution: Notice that $P(x)$ is the longer term in a factorization of $x^{3033} - 1$, where the shorter term is $x^{1011} - 1$. So we are looking for a polynomial which divides the first of these two, but does not divide the second, which would imply that it divides $P(x)$.

There is an easier way to check all of these using complex numbers, which we will cover in a later week. But for now, since this is a multiple-choice question, it suffices to find one that fits the criteria we listed here. It's not immediately clear whether options A,C, or D divide $x^{3033} - 1$, but notice that B is the longer term in $(x^3 - 1) = (x-1)(x^2+x+1)$ and E is the longer term in $(x^9 - 1) = (x^3 - 1)(x^6 + x^3 + 1)$.

We know that both $x^3 - 1$ and $x^9 - 1$ divide $x^{3033} - 1$ since 3033 is divisible by 9, but 1011 is only divisible by 3. This means that $x^{1011} - 1$ is divisible by $x^3 - 1$, but not $x^9 - 1$. But if it was divisible by $x^6 + x^3 + 1$, then it would also have to be divisible by $x^9 - 1$, since $x^9 - 1 = (x^3 - 1)(x^6 + x^3 + 1)$, which is a contradiction. So, option **(E)** divides $x^{3033} - 1$ but not $x^{1011} - 1$, so it must be the answer.

10. **(2022 AMC 10A #21)** There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \cdots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}.$$

What is k?

Solution: We can start by doing the obvious thing, which is writing out the longer term in the factorization corresponding to the left-hand side. It is important to check that we can do this because $289/17 = 17$, which is odd. We get:

$$(2^{17})^{16} - (2^{17})^{15} + (2^{17})^{14} - \cdots - 2^{17} + 1.$$

This is very close to what we want, but we have to manage the negative signs somehow. Notice that the difference between any two powers of 2 is a sum of powers of 2, so we can take pairs and convert them into sums of powers of 2. For example:

$$\begin{aligned}(2^{17})^{16} - (2^{17})^{15} = (2^{17})^{15}\left(2^{17}-1\right) &= (2^{17})^{15}(2-1)\left(2^{16}+2^{15}+\cdots+2+1\right)\\ &= (2^{17})^{15}\left(2^{16}+2^{15}+\cdots+2+1\right)\end{aligned}$$

So, each pair expands into a sum of 17 powers of 2, and there are 8 pairs plus an extra single 1, so our power-of-2 expansion will have $8 \cdot 17 + 1 = 136 + 1 = 137$ total terms, so $\boxed{k = 137}$.

11. **(1992 AIME I #3)** A tennis player computes her "win ratio" by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly 0.500. During the weekend she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than 0.503. What is the largest number of matches that she could have won before the weekend began?

Solution: Let n be the number of matches won before the weekend. This means she must have played exactly $2n$ matches total before the weekend, so after the weekend, she played 4 more games and won 3 more games, and has a win ratio of $\frac{n+3}{2n+4} > \frac{503}{1000}$.

Cross multiplying, $1000n + 3000 > 1006n + 2012$, so $n < \frac{988}{6} = 164\frac{4}{6} = 164\frac{2}{3}$. Thus, the answer is $\boxed{164}$.

12. Find all x such that $-4 < \frac{1}{x} < 3$.

Solution: The tricky thing here is that multiplying by a negative flips the inequality. So we must separately treat the cases where $x > 0$ and $x < 0$.

If $x > 0$, then the first inequality is certainly satisfied since $\frac{1}{x} > 0 > -4$, and we only need $1 < 3x \implies x > \frac{1}{3}$.

If $x < 0$, then the second inequality is certainly satisfied since $\frac{1}{x} < 0 < -3$, and we only need $-4x > 1 \implies x < -\frac{1}{4}$.

So our solutions are given by $x < -\frac{1}{4}$ and $x > \frac{1}{3}$, or in interval notation, $x \in (-\infty, -\frac{1}{4}) \cup (\frac{1}{3}, \infty)$.

13. If $\frac{x^2y}{z} = 24$ and $\frac{y^4z}{x} = 30$, find the value of $\frac{x^8}{(yz)^5}$.

Solution: Notice that the first term gives us a way to change the exponent of y by ±1 and the exponent of z by ∓1 at the same time. The issue with using that on the second term as-is is to get the y and z exponents to be the same is that this always changes the difference between the exponents by 2, but currently, the difference is 3. So, we'll start by squaring the second equation to get a difference of 6:

$$\frac{y^8z^2}{x^2} = 900$$

Then, to make the exponents the same, we want to subtract 3 from the y exponent and add 3 to the z exponent, which we can do by dividing by the first equation 3 times:

$$\frac{y^8z^2}{x^2} \cdot \left(\frac{z}{x^2y}\right)^3 = \frac{y^5z^5}{x^8} = \frac{900}{24^3}$$

This is actually the reciprocal of what we wanted, so our final answer will be:

$$\frac{24^3}{30^2} = \frac{(2^3 \cdot 3)^3}{(2 \cdot 3 \cdot 5)^2} = \frac{(2)(2^3)(2^3 \cdot 3)}{5^2} = \frac{384}{25}$$

14. Let x and y be real numbers satisfying $\frac{2}{x} = \frac{y}{3} = \frac{x}{y}$. Find x^3.

Solution: Our primary goal is to get rid of y. Multiply together the 2nd and 3rd expressions to get $\frac{y}{3} \cdot \frac{x}{y} = \frac{x}{3}$, and notice that this is equal to the 1st expression squared, which is $\frac{4}{x^2}$. So, we have $\frac{x}{3} = \frac{4}{x^2} \implies \boxed{x^3 = 12}$.

15. **(2015 AIME #14)** Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Solution: We can factor the two expressions as:

$$x^4y^5 + y^4x^5 = x^4y^4(x+y) = 810,$$
$$x^3y^6 + y^3x^6 = x^3y^3(x^3+y^3) = x^3y^3(x+y)(x^2+xy+y^2) = 945$$

The first equation gives us $x + y = \frac{810}{x^4y^4}$, which we can plug into the second equation to get

$$\begin{aligned} x^3y^3\frac{810}{x^4y^4}(x^2+xy+y^2) &= 945 \\ \implies \frac{810}{xy}(x^2+xy+y^2) &= 945 \\ \implies \left(\frac{x}{y} + 1 + \frac{y}{x}\right) &= \frac{945}{810} = \frac{7}{6}. \end{aligned}$$

This is a quadratic equation in $\frac{x}{y}$ if we multiply through by $\frac{x}{y}$, so we can solve for xy with the quadratic formula:

$$\left(\frac{x}{y}\right)^2 - \frac{13}{6}\frac{x}{y} + 1 = 0 \implies 6\left(\frac{x}{y}\right)^2 - 13\frac{x}{y} + 6 = 0$$
$$\implies \frac{x}{y} = \frac{13 \pm \sqrt{169-144}}{12} = \frac{13 \pm 5}{12}$$

so $\frac{x}{y} = \frac{3}{2}$ or $\frac{2}{3}$. All the equations we are dealing with are symmetric in x and y (i.e., they treat x and y the same, and wouldn't change if we swapped all the x's and y's), so WLOG, we can take $\frac{x}{y} = \frac{2}{3}$.

We can plug $x = \frac{2}{3}y$ into the second equation, since it looks more similar to the desired expression, and we can use it to solve for y^3 easily. We get

$$\left(\frac{2}{3}\right)^3 y^9 + \left(\frac{2}{3}\right)^6 y^9 = 945 \implies y^9 \left(\frac{2}{3}\right)^3 \left(1 + \left(\frac{2}{3}\right)^3\right) = 945$$

$$\implies y^9 = \frac{945\left(\frac{3}{2}\right)^3}{1 + \left(\frac{2}{3}\right)^3} = \frac{945 \cdot 3^6}{2^3(2^3 + 3^3)} = \frac{27^3}{2^3}$$

$$\implies y^3 = \frac{27}{2}$$

Then, the final expression is:

$$2x^3 + (xy)^3 + 2y^3 = 2\left(\frac{2}{3}\right)^3 y^3 + \left(\frac{2}{3}\right)^3 y^6 + 2y^3$$

$$= 2\left(\frac{8}{27}\right)\frac{27}{2} + \left(\frac{8}{27}\right)\left(\frac{27}{2}\right)^2 + 2 \cdot \frac{27}{2} = 2 \cdot 4 + 2 \cdot 27 + 27 = \boxed{89}.$$