ORMC AMC 10/12 Group
Week 1: Algebra

October 1, 2023

1 Warm-up Exercises

1. Find all x such that $2x^2 + 5 = x^2 + 18$

2. Solve the following system of equations:

$$x - y = 3, \qquad \frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$$

3. Compute $x + y + z$:

$$\begin{aligned} x &= y + z + 2, \\ y &= z + x + 1, \\ z &= x + y + 4. \end{aligned}$$

4. Find all a such that:

$$\frac{3}{1 - \sqrt{a-2}} + \frac{3}{1 + \sqrt{a-2}} = 6.$$

5. Solve for x:

$$\begin{aligned} x + 2y - z &= 5, \\ 3x + 2y + z &= 11, \\ (x + 2y)^2 - z^2 &= 15. \end{aligned}$$

2 Algebraic Manipulations

Useful factoring identities:

- $(a+b)^2 = a^2 + 2ab + b^2$
  - $(x+1)^2 =$ ________________
- $a^2 - b^2 = (a+b)(a-b)$
  - Compute $63 \cdot 65$
  - What is the difference between 2 consecutive perfect squares, i.e. n^2 and $(n+1)^2$?
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
  - Given $a - b = 9$ and $ab = 7$, compute $a^3 - b^3$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  - Notice that $a^3 + b^3 = a^3 - (-b)^3 = (a - (-b))(a^2 + a(-b) + (-b)^2) = (a+b)(a^2 - ab + b^2)$
- $(a+1)(b+1) = ab + a + b + 1$
  - Find all pairs of integers x, y such that $xy + x + y = 9$
  - Find all pairs of integers x, y such that $xy + 3x + 4y = 13$

2.1 Examples

1. Given $3x + \frac{1}{2x} = 3$, find $8x^3 + \frac{1}{27x^3}$.

2. Find all pairs of positive integers a, b which satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$.

2.2 Exercises

1. Find a factorization of $x^5 - y^5$, based on the given factorization for $x^3 - y^3$. Can you find a similar factorization for $x^5 + y^5$? Do the same factorizations work for $x^4 - y^4$ and $x^4 + y^4$?

2. What is the product of all integers x for which $|x^2 - 9|$ is prime?

3. Compute
$$52683 \cdot 52683 - 52660 \cdot 52706.$$

4. **(2008 AMC 10A #7)** Simplify
$$\frac{(3^{2008})^2 - (3^{2006})^2}{(3^{2007})^2 - (3^{2005})^2}.$$

5. **(2021 AMC 12A #9)** Which of the following is equivalent to
$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$
(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

6. **(2008 AMC 12B #16)** A rectangular floor measures a by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

7. Evaluate $2022^3 - 2022^2 \cdot 2023 - 2022 \cdot 2023^2 + 2023^3$

8. **(1987 AIME #5)** Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$

9. **(2022 AMC 12A #21)** Let
$$P(x) = x^{2022} + x^{1011} + 1.$$
Which of the following polynomials is a factor of $P(x)$?
(A) $x^2 - x + 1$ **(B)** $x^2 + x + 1$ **(C)** $x^4 + 1$ **(D)** $x^6 - x^3 + 1$ **(E)** $x^6 + x^3 + 1$

10. **(2022 AMC 10A #21)** There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \cdots < a_k$ such that
$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}.$$
What is k?

11. **(1992 AIME I #3)** A tennis player computes her "win ratio" by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly 500. During the weekend she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than 503. What is the largest number of matches that she could have won before the weekend began?

12. Find all x such that $-4 < \frac{1}{x} < 3$.

13. If $\frac{x^2y}{z} = 24$ and $\frac{y^4z}{x} = 30$, find the value of $\frac{x^8}{(yz)^5}$.

14. Let x and y be real numbers satisfying $\frac{2}{x} = \frac{y}{3} = \frac{x}{y}$. Find x^3.

15. **(2015 AIME #14)** Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.