Olympiads: Warmup

ORMC

10/1/23

Here are some problems to start us off for the year.

1 Warmup Problems

Thanks to Sucharit Sarkar for writing and collecting some of these problems.

Problem 1.1. Use induction to prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Problem 1.2. Use induction to prove that $2!4!\dots(2n)! \geq ((n+1)!)^n$

Problem 1.3. Let x be a real number. Show that there is a number among $x, 2x, \dots, (n-1)x$ that differs at most $\frac{1}{n}$ from an integer.

Problem 1.4. Show that every positive integer can be written as a sum of distinct Fibonacci numbers.

Problem 1.5 (USSR Olympiad Problem Book, 104)**.** Does there exist a natural number n such that the fractional part of the number $(2+\sqrt{2})^n$, that is, the difference

$$(2+\sqrt{2})^n - \left\lfloor (2+\sqrt{2})^n \right\rfloor,$$

is more than 0.999999?

Problem 1.6 (USSR Olympiad Problem Book, 217)**.** Prove that if the product of two polynomials with integer coefficients is a polynomial with even coefficients, not all of which are divisible by 4, then in one of the polynomials, all the coefficients are even, while in the other, not all the coefficients are even.

2 Competition Problems

Problem 2.1 (CHMMC 2016, Team Round 9)**.** Find the sum of all 3-digit numbers whose digits, when read from left to right, form a strictly increasing sequence. (Numbers with a leading zero, e.g. "087" or "002", are not counted as having 3 digits.)

Problem 2.2 (CHMMC 2016, Team Round 8)**.** Let n be a positive integer. If S is a nonempty set of positive integers, then we say S is n-complete if all elements of S are divisors of n, and if d_1 and d_2 are any elements of S, then $\frac{n}{d_1}$ and $\gcd(d_1, d_2)$ are in S. How many 2310-complete sets are there?

Problem 2.3 (BAMO 2013 Problem 4)**.** For a positive integer $n > 2$, consider the $n-1$ fractions

$$\frac{2}{1}, \frac{3}{2}, \ldots, \frac{n}{n-1}.$$

The product of these fractions equals n, but if you reciprocate (i.e. turn upside down) some of the fractions, the product will change. Can you make the product equal 1? Find all values of n for which this is possible and prove that you have found them all.

Problem 2.4 (BAMO 2011 Problem 2)**.** Five positive integers are arranged in a row. The integers must be chosen such that the sum of the digits of the neighbor(s) of a given circle is equal to the number labeling that point. In the example

$$18, 23, 59, 22, 46,$$

the second number $23 = (1+8) + (5+9)$, but the other four numbers do not have the needed value.

What is the smallest possible sum of the five numbers? How many possible arrangements of the five numbers have this sum? Justify your answers.

Problem 2.5 (USAMO 2002 Problem 5)**.** Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \le i < k)$.

Problem 2.6 (USAMO 2006 Problem 5)**.** A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \ge 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

Problem 2.7 (Putnam 2013 Problem A1)**.** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Problem 2.8 (Putnam 2022 B3)**.** Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?