Nimbers

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1 Impartial Games

Definition 1. A *combinatorial game* is a game that is

- two-player (two players alternate turns)
- complete-information (no face-down cards)
- deterministic (no randomness)
- well-founded (it's guaranteed to finish in a finite amount of turns)
- where the first player who cannot move loses, and thus the other player wins

We will start by studying *impartial* combinatorial games, which are games where the rules are the same for each player. It may still be better to be the player who goes first or second. We will omit the word "combinatorial" because all the games we consider in this packet are combinatorial games, so "impartial games" refer to impartial combinatorial games.

Problem 1. Here are some familiar examples of games. Do these meet the definitions, and are they impartial?

- Tic-tac-toe
- Chess
- Checkers
- Poker
- Football (whatever that means to you)

When we study games, we will think about *positions* - a position is the state of the game, recording the possible moves for each player, right before a particular turn.

In an impartial combinatorial game, a position is called a winning position or first-player win if the first player is guaranteed to win if they use the right strategy, and a losing position or second-player win if the second player is guaranteed to win if they use the right strategy.

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1.1 Nim

Perhaps the most classic impartial game to study is Nim. There are a few piles of objects (for our purposes, a few different groups of toothpicks on the table). On each turn, you take a positive number of toothpicks from *exactly one* pile. Play proceeds until someone empties the last pile, and thus wins, because there is no valid move for the other player.

Problem 2. Use the toothpicks to play games of Nim. Start with 3 piles where the first pile has 5 toothpicks, the second pile has 6 toothpicks, and the third pile has 7 toothpicks. If your group has some people who have not played Nim before, they should try to play each other first, and then face more experienced players.

Problem 3. Say we're playing Nim with two piles - with m and n toothpicks respectively (m and n could be zero, in which case there are really 0 or 1 piles). For what values of m and n is this a winning or losing position?

Problem 4. Say you have an impartial game, and an algorithm for determining whether any given position is a winning or losing position. Describe a winning strategy for the game, assuming you get to choose whether you go first or second at the beginning.

Problem 5. Prove that each position in an impartial combinatorial game is guaranteed to either have the first or second player win - and is thus either a winning or a losing position. (Hint: If you have a game where every move has a winning strategy, show that the game itself has a winning strategy. Then if you have a game without a winning strategy, find a contradiction to well-foundedness.)

1.2 Other Subtraction Games

Subtraction games are variations on Nim - still impartial games. Let S be a set of positive integers. Then once again let us populate some piles with toothpicks, and once again you remove them from one pile at a time, but now you are only allowed to move a number of toothpicks in S.

Problem 6. (a) Let $S = \{1, 2\}$. Set up the game so that pile 1 has 5 toothpicks, pile 2 has 6 toothpicks, and pile 3 has 7 toothpicks.

(b) Let $S = \{1, 4\}$. Set up the game so that pile 1 has 5 toothpicks, pile 2 has 6 toothpicks, and pile 3 has 8 toothpicks.

(c) Let $S = \{1, 4\}$. Set up the game so the only pile has 5 toothpicks.

Feel free to mix and match S with the configuration, and generally play games as much as you want until you get a feel for the strategy.

1.3 Adding Games Together

I want to teach you a new board game. It's called *Chess + Checkers*. To play, we sit down on opposite sides of a table, and set up a chessboard and a checkerboard side-by-side. On each of our turns, we pick one of the boards and do a move in that board. As if chess and checkers aren't hard enough (well, maybe checkers isn't hard enough), now you have to decide whether to respond to my threat in chess, or get ahead in checkers.

In general, we can add any two combinatorial games G_1, G_2. In the new game, $G_1 + G_2$, each player gets to move in only one game at a time, and the person who runs out of moves in *both* games loses. (That is, if G_1 is finished first, no matter who won G_1, the winner of G_2 is the overall winner.)

Problem 7. Convince yourself that the sum of two impartial games is impartial, and the sum of two games of Nim is also a game of Nim (how many piles are there? of what sizes?).

Problem 8. Is this addition commutative? Is it associative? (Reminder: Commutativity means that $a+b=b+a$ for all a, b. Associativity means that for all a, b, c, we have $(a+b)+c=a+(b+c)$.)

> **Solution.** Yes, and yes. If you imagine putting multiple game boards on the same table, that it doesn't matter what order you do that in.

Problem 9. Let G be an impartial game. Who wins $G+G$?

> **Solution.** This is a losing position, so the second player wins, by always mimicking the first player's move on the other copy of G, so both games stay in the same position, until eventually the first player finishes one game, and the second player finishes the other.

1.4 Sprague-Grundy Values

Definition 2. Let G be an impartial game with only finitely many valid moves at each position (like, for instance, Nim with finitely many toothpicks). Define the *Sprague-Grundy* value of G (written $\mathrm{SG}(G)$) recursively as follows:

- The empty game (with no valid moves) has a Sprague-Grundy value of 0.
- Calculate the Sprague-Grundy value of each position you can legally move to. Then let the Sprague-Grundy value be the smallest natural number that isn't on that (finite) list. We take the natural numbers to include 0 in this worksheet.

Problem 10. For the empty game, who has the winning strategy and why?

> **Solution.** The second player has the winning strategy. The first player has no legal moves and, thus, loses.

Problem 11. For a natural number n, let $*n$ represent a Nim game with one pile of n toothpicks (so $\mathrm{SG}(*0)=0$). What is the Sprague-Grundy value of $*n$?

Solution. Show by induction that the S-G value of $*n$ is n. The set of allowed moves is $\{*0, *1, *2, \ldots, *(n-1)\}$, the set of disallowed S-G values will just be $\{0, 1, 2, \ldots, n-1\}$, so the smallest number not on the list will be n.

Problem 12. Let's try a different Nim-ish game. There is one pile of n toothpicks, and at each turn, you must remove either 1 or 4 toothpicks. What is the Sprague-Grundy value of this game for each n? For which values of n is this a winning or losing position?

Solution. If we calculate the first few values, we get $0, 1, 0, 1, 2$. These five values repeat forever, and we can prove this by induction.

Problem 13. Prove that an impartial game is a second-player win if and only if its S-G value is 0.

Solution. Assume that the S-G value of an impartial game is greater than 0. Then the first player can continue to make moves that keep the S-G value of 0. In finitely-many moves, the first player will get to the empty game.

Assume an impartial game is a first-player win. Then there is some sequence of moves so that the game reaches the empty game after the first player's move. Thus 0 will be in the disallowed list and the S-G value of the game will be positive.

This even gives us a winning strategy! For finite-move impartial games, we can recursively determine any position's Sprague-Grundy value, and then we know whether that position is a winning or losing position! Then it's just a matter of trying to move to losing positions whenever possible.

1.5 Nim Addition

To calculate the Sprague-Grundy values of Nim games, and thus to develop a winning strategy for Nim, we will define a special kind of addition on the natural numbers, called *Nim-addition*, and written with $\oplus$. We define it so that $m \oplus n = \mathrm{SG}(*m + *n)$, that is, to add m and n this way, we find the Sprague-Grundy value of a Nim game with one pile of height m and one pile of height n. It turns out that for any impartial games G, H, $\mathrm{SG}(G + H) = \mathrm{SG}(G) \oplus \mathrm{SG}(H)$. If you are interested, you can find a proof of this (called the Sprague-Grundy theorem) on Wikipedia: `https://en.wikipedia.org/wiki/Sprague%E2%80%93Grundy_theorem`.

Problem 14. Calculate a Nim-addition table for the numbers $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

Solution. You can find the table here: `https://en.wikipedia.org/w/index.php?title=Nimber&oldid=383699838`

Problem 15. Form a guess for calculating $m \oplus n$ in general.

Hint: Use your table for inspiration, and try writing m and n in binary.

Solution. The answer is that if you have just two piles, you can find the value of $*m + *n$ by writing m and n in binary, and adding/XORing bitwise.

Problem 16. Go back to your earlier Nim games, and see if you can use Nim-addition to win!

Problem 17. Prove that your formula for $m \oplus n$ works.

Hint: Use a kind of induction. Assume that this formula works for calculating $a \oplus b$ whenever $a + b < 2^m + 2^n$. Then check that $2^m \oplus 2^n$ follows your formula.

Solution. Hint for proving this: Assume $m < n$ and that n is a power of two. Then show that $m \oplus n = m + n$. (Assume for induction that the whole binary principle is true for all numbers less than n.)

Then you can express any number as a binary expansion, and you get $2^{i_0} \oplus 2^{i_1} \oplus \cdots \oplus 2^{i_k} = 2^{i_0} + 2^{i_1} + \cdots + 2^{i_k}$. Then you can use associativity and the fact that $m \oplus m = 0$ to get the result.

Problem 18. For what values of n is the set $\{0, 1, 2, \ldots, (n-1)\}$ closed under Nim-addition? (That is, if $i, j < n$, then $i \oplus j = k$ for some $k < n$.)

Solution. This is true iff n is a power of 2. If $n = 2^a$, then $i, j < n$ iff i, j can be written with a bits, so their binary addition will also have at most a bits.

If n is not a power of 2, let 2^a be the largest power of 2 less than n. Then $2^a, 2^a - 1 < n$, and $*2^a + *(2^a - 1) = *(2^a + (2^a - 1)) = *(2^{a+1} - 1) \geq n$, so the set is not closed.

1.6 Bonus

Problem 19. Let's look at a modification of the earlier subtraction problem. There is one pile of n toothpicks, and at each turn, you must remove a number of toothpicks which is a power of 4. What is the Sprague-Grundy value of this game for each n? For which values of n is this a winning or losing position?

Solution. If we calculate the first few values, we get $0, 1, 0, 1, 2$. These five values repeat forever, and we can prove this by induction, using the fact that every power of 4 is either 1 or 4 mod 5. This means that this is fundamentally the same game as where you can only move 1 or 4 pieces!