2 Euler's Product Formula for Sine

Recall that if $p(x)$ is a polynomial with the leading term x^n, then $p(x) = (x-x_1)(x-x_2)\dots(x-x_n)$ where $x_1, \dots, x_n$ are the roots of $p(x)$.

Problem 11. Prove that if $p(x)$ is a polynomial with $p(0) = 1$, then $p(x) = \left(1 - \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right)\dots\left(1 - \frac{x}{x_n}\right)$, where x_i are inverses of roots of $p(x)$.

We will attempt to extend this decomposition to an infinite degree function $\mathrm{sh}x = \frac{e^x - e^{-x}}{2}$ with roots $0, \pm i\pi, \pm 2i\pi, \dots$.

Theorem 1.

$$\frac{\mathrm{sh}x}{x} = \prod_{k=1}^{\infty}\left(1 + \frac{x^2}{k^2\pi^2}\right)$$

Recall that $a^n - b^n = \prod_{k=0}^{n-1}(a - e^{\frac{2\pi i k}{n}}b)$

Problem 12. Factorize the polynomial $\left(1 + \frac{x}{n}\right)^n - \left(1 - \frac{x}{n}\right)^n$ into linear factors.

Definition 2. The sum (product) $a_1 + a_2 + a_3 + \dots$ $(a_1a_2a_3\dots)$ is called *convergent*, if for all hypernaturals $N < M$, the sum $a_{N+1} + a_{N+2} + \dots a_M \sim 0$ $(a_{N+1}a_{N+2}\dots a_M \sim 1)$.

Problem 13. Prove that a product $(1+a_1)(1+a_2)\dots$ is convergent if and only if the sum $a_1+a_2+\dots$ is covergent.

Problem 14. Prove that the product $\prod_{k=1}^{\infty}\left(1 + \frac{x^2}{k^2\pi^2}\right)$ is convergent.

Problem 15. (Robinson's lemma) Let $f(x,y)$ be a real function, and let Y be a hyperreal. Prove that if $f(n,Y)$ is infinitesimal for all standard naturals n, then there is a hypernatural H such that $f(n,Y)$ is infinitesimal for all hypernaturals $n \leq H$.

Problem 16. Use Robinson's lemma to finish the proof of the theorem.

Problem 17. Prove the formula

$$\pi \coth \pi x = \sum_{k=-\infty}^{\infty} \frac{1}{x-k}$$

(Hint: differentiate Euler's formula)

3 Application of Euler's formula

Historically, the reason Euler invented this formula was to solve the *Basel's problem*: find the sum of inverse squares $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$

Problem 18. Solve the Basel's problem by equating coefficients in x^3 in two sides of Euler's product formula. Can you find the sum of inverse fourth powers?

Definition 3. We define the n-th *Bernoulli number* B_n to be the coefficients in the expression $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$.

Problem 19. Prove that $B_1 = -\frac{1}{2}$ but $B_k = 0$ for all other odd k. (Hint: a function $f(t) = \sum a_n t^n$ has no odd powers of t if it is even)

Problem 20. Show that, in general, $1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \ldots = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}$. (Hint: expand every $\frac{1}{x-n}$ in $\pi \cot \pi x$ in a geometric series)

In general, the function $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$ is called the *Riemann zeta function.* Apery proved in 1978 that $\zeta(3)$ is irrational. It is a notorious open problem to prove that all the others odd zeta values $\zeta(5), \zeta(7), \ldots$ are irrational.

Problem 21. Prove that $\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$, the product taken over all the prime numbers.

Problem 22. What is the probability that the two randomly chosen naturals are coprime?