DIVIDING ZERO WITH THE SLEEPWALKING BISHOP

JUNIOR CIRCLE 10/14/2012

1. Warm Up:A quick review of modular arithmetic

Problem 1. Fill in the blank space!

(a) $3 + 9 \equiv \quad \mod 10$

(b) $5 \times 2 \equiv \quad \mod 7$

(c) $11 + 11 \equiv \quad \mod 2$

Problem 2 (Take mods, then Compute!)**.** I would suggest you follow the hint– again, fill in the blank.

(1)
$$13684735 \times 163873882794 \mod 2$$

(2)
$$26487739 + 3687287304 \mod 10$$

(3)
$$247980 \times 19378469280 \mod 10$$

Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

2. Division $\mod n$

What does it mean for a whole number a to divide another number b? It means that there is a third number x, such that

$$a \times x = b$$

For example, 3 divides 12 because there exists 4 so that

$$3 \times 4 = 12$$

Finding the answer to the problem $12 \div 3$ is equivalent to solving the equation

$$3 \times x = 12$$

...

We have a good definition for divisibility of whole numbers. We want to use this definition for division in $\mod n$ arithmetic. The way to change our old definition for divisibility of whole numbers into one for $\mod n$ is to replace the equal sign with a congruence sign and stick on a $\mod n$ at the end. We will say that a number a divides a number $b \mod n$ if there exists a number x that makes the following equation true

$$a \times x \equiv b \mod n$$

Here are a few examples of divisibility $\mod 10$.

- We have that 2 divides $6 \mod 10$, as $2 \times 3 \equiv 6 \mod 10$
- We have that 3 divides $7 \mod 10$, as $3 \times 9 \equiv 27 \equiv 7 \mod 10$
- We have that 3 divides $5 \mod 10$, as $3 \times 5 \equiv 15 \equiv 5 \mod 10$
- We have that 5 divides $0 \mod 10$, as $2 \times 5 \equiv 10 \equiv 0 \mod 10$

Problem 3. Rewrite the following equations using a division sign and solve.

(1) $5 \times x = 25$

(2) $7 \times x = 63$

Problem 4. First construct a multiplication table for $\mod 5$, then do the following divisions $\mod 5$.

$\times_5$	0	1	2	3	4
0					
1					
2					
3					
4					

Rewrite each division problem as a equation, and then solve the equation. For example, if the division was

$$3 \div 2$$

we first write

$$2 \times x \equiv 3 \mod 5$$

and then solve for x.

(a) $4 \div 1 \equiv \quad \mod 5$

(b) $4 \div 3 \equiv \quad \mod 5$

(c) $3 \div 4 \equiv \quad \mod 5$

(d) $3 \div 2 \equiv \quad \mod 5$

Problem 5 (A method for Division using multiplication tables)**.** Let's find a faster way to do division using Multiplication tables

(a) Let's say that I wanted to find $30 \div 6$. Describe how I would find the solution using this multiplication table.

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	8	10	12
3	0	3	6	9	12	15	18
4	0	4	8	12	16	20	24
5	0	5	10	15	20	25	30
6	0	6	12	18	24	30	36

(b) Let's say that I wanted to tell if a was divisible by b. How could I use the multiplication table to quickly find this out? What if a was divisible by b, how would I find $a \div b$?

Problem 6 (Division using whole numbers is unique)**.** . Can you find a different number greater than 0 so that

$$2 \times c = 3$$

Explain.

Problem 7 (Division is *not* unique $\mod 12$)**.** (a) Show that 2 divides 6 in $\mod 12$ arithmetic by finding a number b greater than 0 and less than 12 so that

$$2 \times b \equiv 6 \mod 12$$

(b) Can you find another number c, so that

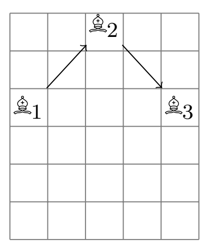
$$2 \times c \equiv 6 \mod 12$$

(c) With this in mind, does $6 \div 2$ make any sense $\mod 12$?

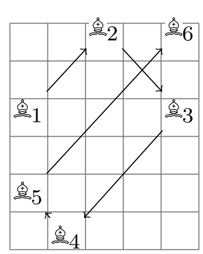
3. The Sleepwalking Bishop

The **Bishop** is a piece from the game of chess that can move diagonally. We draw the bishop with the symbol ♗.

The Bishop of Bounce lives in courtyard that is covered in little squares. His yard measures 5 by 6 squares. The bishop is a sleepless fellow, and sometimes in the middle of the night he wakes up and begins to stumble around his yard. As he sleepwalks, he walks in a straight line and when he reaches a wall, he bounces off the wall and keeps on walking. For instance, one possible route the bishop may start on a particular night is



The bishop goes back to sleep when he walks into a corner. So, the full walk that the bishop would take on the night above would be

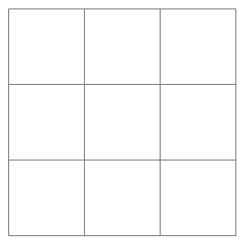


Problem 8. Can the Bishop ever visit every square of the courtyard? Why or why not?

Problem 9. Where should the bishop start if he wants to visit as many squares as possible when sleepwalking?

Problem 10. Is there a place that the bishop can start so never runs into a corner and falls asleep? (hint! Use the previous problem)

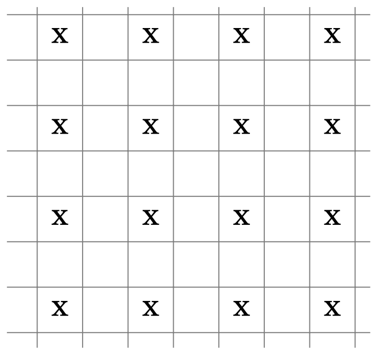
The Bishop of Bounce has a good friend, the Bishop of Tic-Tac-Toe. The Bishop of Tic-Tac-Toe has a courtyard that is 3 squares long by 3 squares wide.



Problem 11. Why does the Bishop of Tic-Tac-Toe sleepwalk considerably more than the Bishop of Bounce on some nights?

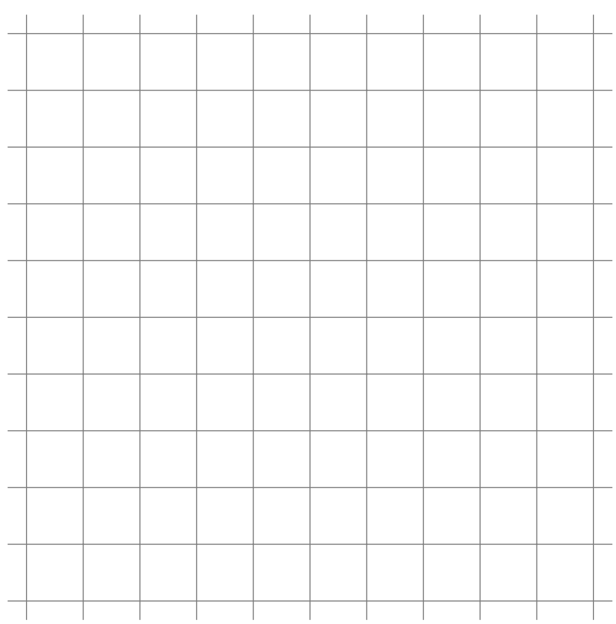
Problem 12. What it is the shortest sleepwalk that the Bishop of Tic-Tac-Toe take?

Last year, the Bishop of Bounce and Bishop and the Bishop of Tic-Tac-Toe had a friend that passed away, the late Bishop Walkthroughwalls. His ghost now haunts the Bishop of Bounce and his neighbor's yard! If left unstopped, he will wander the world forever. His world is like one infinitely large grid.
In order to help their friend, the Bishop of Bounce and Bishop of Tic-Tac-Toe place markers for their dead friend to rest peacefull as shown on the picture.



Problem 13. Is there a starting place on this grid which will allow the Bishop Walkthroughwalls to float on forever? How is this similar to the situation that Bishop Tic-Tac-Toe faced in his garden?

Problem 14. The Bishop of Bounce realizes there is a problem with the layout of x's . As the Bishop of Bounce is cheap, he doesn't want to place to crosses next to eachother. How can he place marks across the infinite grid so that the late Bishop Walkthroughwalls will find a resting place no matter where he starts?



4. Primes Numbers and Zero Divisors

You may have heard of prime numbers before. What is a prime number? We will use this definition:

Definition 1. We call a whole number p **prime** if the only numbers that divide p are 1 and p.

What are a few examples of prime numbers? Well, $2, 3, 5, 7, 11 \ldots$ are a few examples of prime numbers that you may have heard of.

We also want to talk about when a non-zero numbers divide 0! Here are some examples:

- 4 divides $0 \mod 10$ because $4 \times 5 \equiv 20 \equiv 0 \mod 10$
- 2 divides $0 \mod 10$ because $2 \times 5 \equiv 10 \equiv 0 \mod 10$
- 5 divides $0 \mod 10$ because $5 \times 8 \equiv 40 \equiv 0 \mod 12$

We say that if $a \times b \equiv 0 \mod n$, and neither a or b is congruent to 0 $\mod n$, then we call a a **zero divisor**. So $2, 4$ and 5 are all examples of zero divisors $\mod 10$

Problem 15. Check if the following numbers divide $0 \mod 12$

(a) Does 4 divide 0 in $\mod 12$?

(b) Does 7 divide 0 in $\mod 12$?

(c) Does 2 divide 0 in $\mod 12$?

(d) Does 3 divide 0 in $\mod 12$?

Problem 16. Find all the zero divisors between 1 and 5 for multiplication $\mod 6$

Problem 17 (Not-Unique Division and Zero Divisors)**.** We show that if you have non-unique division, you can find a zero divisor

(a) Find a number x between 0 and 9 so that

$$2 \times x \equiv 4 \mod 10$$

(b) Find a different number y between 0 and 11 so that

$$2 \times y \equiv 9 \mod 10$$

(c) Subtract the first congruence from the second one. Conclude that 2 is a zero divisor in $\mod 10$ arithmetic

Problem 18 (Zero Divisors and Not-Unique Division)**.** We show that if you have a zero divisor, then division is not unique.

(a) Find a number z between 1 and 9 so

$$5 \times z \equiv 0 \mod 10$$

(b) Find a number x between 1 and 9 so that

$$5 \times x \equiv 5 \mod 10$$

(c) Add the two congruences together to conclude that there is a number $y \neq x$ so that

$$5 \times y \equiv 5 \mod 10$$

and conclude that division is not unique $\mod 12$

Problem 19 (Division is unique $\mod 5$)**.** Construct a multiplication table for $\mod 5$

$\times_5$	0	1	2	3	4
0					
1					
2					
3					
4					

If a divides $c \mod 5$, then there is a number b so that

$$a \times b \equiv c \mod 5$$

Show that if a divides $c \mod 5$, then there is **unique** b so that

$$a \times b \equiv c \mod 5$$

(Hint! Use the multiplication table above, as well as the method for finding division that you came up with in Problem 5)

Problem 20 (Finding a number with a zero divisor)**.** Here we will be using the definition of prime to help us. Recall that if n is *not* prime, it can be written as $a \times b$ for some numbers a and b, neither of which are 1.

(a) Show that a is a zero divisor $\mod n$

(b) Conclude that if a number is *not* prime, it has a zero divisor.

Problem 21. Recall from last week that we call a multiplication table **Sudoku** if it contains every number once in each row and column (besides the 0 row and column).

(a) Why is it if a $\mod p$ multiplication table is Sudoku, then there only one solution to $a \div b \mod n$? (Hint: Think about Problem 5)

(b) Why does it follow that if a $\mod p$ multiplication table is Sudoku, there are no zero divisors?

Problem 22 (A return to the sleeping Mathematicians)**.** Jonathan, Isaac, Derek, Jeff and Morgan have gone to vacation in outer space!

(a) Derek plans on visiting the planet Mars. On Mars, they use clocks that have 7 hours on them. Derek decides on the following macabre sleep pattern: When he wakes up, he will look at his clock. He will then take whatever number he sees, and sleep for that many hours. If he wakes up at 7 oclock, he will get out of bed. Why is this a terrible idea?

(b) Isaac is on the planet Venus. On Venus, the clocks are only 6 hours long. Every time Isaac wakes up, he looks at his clock. If he has not seen that number before, he goes back to sleep. If Isaac wants to see every number on his clock, how long should each one of his naps be? (note: a proper nap is longer than 1 hour)

(c) Morgan is on the planet Jupiter. As it is quit a bit bigger than Pluto, it's day is 11 hours long. Morgan is like Derek, and is a bit rocket-lagged from his travels. When he wakes up, he looks at his clock, and sleeps for 3 times the amount that appears on his clock. Assuming Morgan starts at 2 o'clock, Does Morgan ever look at his clock and see 11 o'clock? What about 1 o'clock?