Nonstandard analysis

Nikita and Stepan

Summer 2023

The conduit is available at `https://tinyurl.com/ORMCconduit`.

1 Exponential function

Imagine you have a very peculiar bank, and you deposit \$1 in an account that promises a 100% annual interest rate. But rather than compounding this interest once at the end of the year, the bank offers different compounding options:

- **Annually:** If compounded annually, you'd get your interest at the end of the year. So, after one year, you'd have \$2 (\$1 initial deposit + \$1 interest).
- **Semiannually:** Now, if the bank compounded your interest semi-annually (every 6 months), you'd first get 50% after 6 months, turning your \$1 into \$1.50. Then, you'd get another 50% interest on the \$1.50 after another 6 months, giving you \$2.25 by the end of the year.
- **Quarterly:** If compounded quarterly, you'd get 25% every 3 months. This would make your money grow even more by the end of the year.
- **Monthly:** With monthly compounding at a rate of $\frac{100\%}{12} = 8.33\%$ per month, the final amount increases even more.
- **Daily:** Here, you'd be earning interest every day at a rate of $\frac{100\%}{365}$. Your payoff will be \$2.714567...

... And so on.

Problem 1. What is your final payoff if you compound interest 100 times per year?

Problem 2. Which is better: 20% annual interest rate compounded monthly or 25% annual rate compounded semiannually?

If you compound interest every $1/H$ of the year where H is an unlimited hypernatural number, your payoff is equal to $\left(1+\frac{1}{H}\right)^H$. The standard part of this number is denoted by e and is equal to 2.71828...

We are leaving open the question for now why this definition doesn't depend on H.

Problem 3. a) Prove that for $a > 1$ one has $a^n - 1 > n(a-1)$.

b) Prove that for $a > 1$ one has $a^{\frac{1}{n}} - 1 < \frac{a-1}{n}$.

c) Prove inductively that for $x > -1$ one has $1 + nx \leq (1+x)^n$. This is known as *Bernoulli's inequality*.

We know what is a^b for positive real a a rational b (say $\pi^{\frac{2}{3}} = \sqrt[3]{\pi^2}$). Extending it to real b requires some analysis.

Definition 1. Let $\frac{H}{K}$ be a hyperrational number infinitesimally close to b. Then $a^b = \text{st}\sqrt[K]{a^H}$.

Problem 4. Prove that for two infinitesimally close rational numbers $r = \frac{H}{K}$ and $r + \varepsilon = \frac{M}{N}$ one has $a^r \sim a^{r+\varepsilon}$ for every standard $a > 1$. Then prove it for $0 < a < 1$. (Hint: use part b from the previous problem)

This problem proves that a^b is well-defined. Now we start proving that e is well-defined.

Problem 5. a) Prove that $(1 + \frac{1}{n})^n < 1 + 1 + \frac{1}{2} + \frac{1}{3!} + ... + \frac{1}{n!}$ for integer n.

b) Prove that $(1 + \frac{1}{N})^N$ is limited for any hypernatural N.

c*) Prove that $(1 + \frac{1}{N})^N \sim 1 + 1 + \frac{1}{2} + \frac{1}{3!} + ... + \frac{1}{N!}$

d*) Prove that $e_N = \text{st}(1 + \frac{1}{N})^N$ is irrational.

Note that we do not yet know that e_N is the same number for all N.

Problem 6. Assume that $f(x) = a^x$ is differentiable at $x = 0$. Prove that then $f(x)$ is differentiable at any point and $f'(x) = a^x f'(0)$.

Problem 7. a) Use Bernoulli's inequality to prove $e_N^x \geq 1 + x$ for all real x;

b) Prove that the angle between vectors $(x, e_N^x - 1)$ and $(-x, 1 - e_N^{-x})$ is $\sim 180°$ for $x \sim 0$;

c) Deduce that line $y = 1 + x$ is tangent to the graph $y = e_N^x$;

d) Prove that $e_N = e_M$ for all unlimited hypernaturals N,M.

Problem 8. Prove that $(a^x)' = \ln(a)a^x$, where $\ln(a)$ is logarithm base e. (Hint: what is $(e^{bx})'$?)

Problem 9. Find the derivative of x^x.

Problem 10. Which of the following numbers are limited? Find standard parts of those that are:

$$H^{-H}, \frac{4^\varepsilon - 4^{-\varepsilon}}{2^\varepsilon - 2^{-\varepsilon}}, 3^H - H2^H, \ln(H+1) - \ln(H), 8^{\sqrt{H}} - 2^H, \cos(\varepsilon)^{\varepsilon^2}.$$