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The 15 Puzzle, Part II

Problem 1 *Find the order of the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$.

Problem 2 *Without doing any more computations, find the following for the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ *from Problem 1.*

$$\sigma^{-1} =$$

$$\sigma^{126} =$$

1 What's Your Sign?

If a permutation σ moves the element in the position i to the position k, we write $\sigma(i) = k$. Let us consider the permutation $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ one more time. It moves the fifth element to the second position, so $\sigma(5) = 2$. It moves the first element to the fifth position, so $\sigma(1) = 5$.

Problem 3 *For the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$, *find the following.*

$$\sigma(2) =$$

$$\sigma(3) =$$

$$\sigma(4) =$$

If $i < j$, but $\sigma(i) > \sigma(j)$, then the pair (i, j) is called an *inversion* of the permutation σ. In other words, a inversion of a permutation is a smaller number moved to the right of a larger number (or a larger number moved to the left of a smaller number). For example, the permutation $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ moves 5 to the second position, so $(5, 1)$, $(5, 4)$, and $(5, 3)$ are all inversions of σ.

Note 1 *Although the words "inverse" and "inversion" are very similar, the notions of an inverse of a permutation and an inversion of a permutation are very different! An inverse of a permutation σ is the permutation σ^{-1} that undoes what the original permutation σ does. The inversion of a permutation σ is a disorder the permutation σ creates.*

Problem 4 *Write down all other inversions of the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$.

The *sign* of a permutation is defined according to the following formula.

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$$

where $N(\sigma)$ is the number of inversions of the permutation σ. For example, the total number of inversions of the permutation $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ is seven (check it!), so $\operatorname{sgn}(\sigma) = (-1)^7 = -1$.

Problem 5 *What is the sign of a trivial permutation?*

$$sgn(e) =$$

Problem 6 *Find the signs of the following permutations.*

$$sgn\begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} =$$

$$sgn\begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix} =$$

Problem 7 *What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**		**11**
13	**15**	**14**	**12**

Problem 8 *What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)*

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

Problem 9 *Write down all the inversions of the permutation* $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$.

What is the sign of the permutation?

$$sgn(\sigma) =$$

2 Parity of a Permutation

Recall that a transposition (ji) is a permutation that changes the positions of only two elements, i-th and j-th.

Theorem 1 *The sign of any transposition is* -1*.*

Before giving Theorem 1 a formal proof, let us check a few cases.

Problem 10 *What is the sign of the transposition* $\sigma = (52)$ *acting on a set of five elements?*

$$sgn(\sigma) =$$

What is the sign of the transposition $\sigma = (52)$ *acting on a set of six elements?*

$$sgn(\sigma) =$$

What is the sign of the transposition $\sigma = (63)$ *acting on a set of seven elements?*

$$sgn(\sigma) =$$

To prove Theorem 1, let us first observe that a transposition of two neighbouring elements, called an *adjacent transposition*, always changes the number of inversions by one. Let us consider the transposition $\delta = (i+1, i)$. All the elements except for the $i+1$-st that formed inversions with the i-th element still form inversions with it when it moves to the $i+1$-st position. All the elements except for the i-th that formed inversions with the $i+1$-st one keep doing so when the latter moves one position to the left. If the pair $(i, i+1)$ formed an inversion, δ removes it. If the pair formed no inversion, δ creates one.

The following lemma finishes the proof of Theorem 1.

Lemma 1 *Any transposition can be realized as a product of an odd number of adjacent transpositions.*

Proof — Consider the transposition (ji) where $j > i+1$. The following product of $j-i-1$ adjacent transpositions

$$(j-1, j-2) \circ \ldots \circ (i+2, i+1) \circ (i+1, i)$$

moves the i-th element to the $j-1$-st position one step at a time. The adjacent transposition

$$(j, j-1)$$

swaps it with the j-th element. Finally, the following product of $j-i-1$ adjacent transpositions

$$(i+1, i) \circ (i+2, i+1) \circ \ldots \circ (j-1, j-2)$$

moves the element that was originally in the j-th position to the i-th. This way, any transposition (ji) where $j > i+1$ can be represented as a product of $2(j-i-1)+1$ adjacent transpositions. □

Example 1

$$(52) = (32) \circ (43) \circ (54) \circ (43) \circ (32)$$

Problem 11 *Represent the transposition* (63) *as a product of adjacent transpositions.*

$$(63) =$$

Is the number of the adjacent transpositions odd or even?

Problem 12 *Represent the transposition* (41) *as a product of adjacent transpositions.*

$$(41) =$$

Is the number of the adjacent transpositions odd or even?

The permutations that have the sign 1 are called *even*. The permutations that have the sign -1 are called *odd*. This way, all permutations are split into two classes. A class of a permutation is called its *parity*. Theorem 1 proves that transpositions are odd permutations and that multiplying a permutation by a transposition changes the parity of the former.

Problem 13

- *Find the sign of the permutation* $\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$ *acting on a set of five elements.*

$$sgn(\mu) =$$

- *Find the product* $(51) \circ \mu$.

$$(51) \circ \mu =$$

- *Find the sign of the permutation* $(51) \circ \mu$.

$$sgn\left((51) \circ \mu\right) =$$

Note that Theorem 1 gives a different way to compute the sign of a permutation. Instead of counting inversions, let us decompose the permutation into a product of transpositions. Then the sign of the transposition is

$$(-1)^{\mathrm{n}}$$

where n is the number of transpositions in the product. Various representations of a permutation as a product of transpositions can have different length, but they always have the same parity.

Problem 14 *Represent the permutation* $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$ *from Problem 9 as a product of transpositions.*

$$\sigma =$$

Use the formula

$$sgn(\sigma) = (-1)^n$$

where n *is the number of transpositions in the product to find the sign of* σ*. Compare your answer to that of Problem 9.*

$$sgn(\sigma) =$$

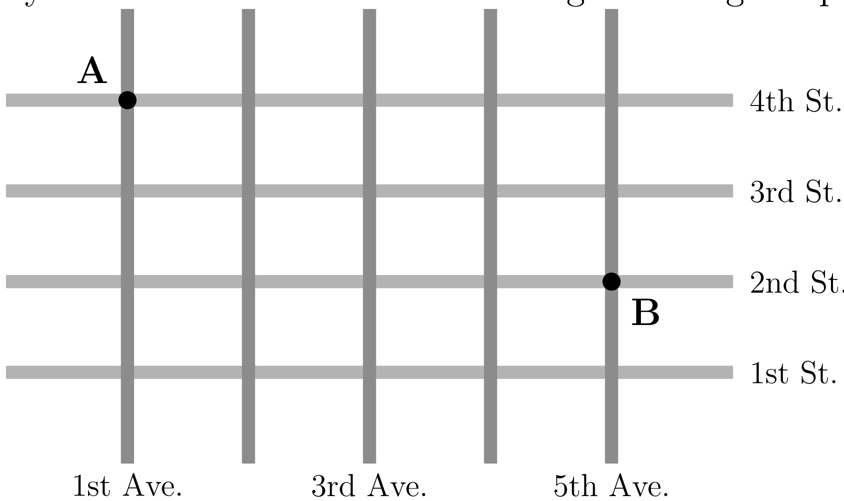
Every move of the 15 puzzle is a transposition of a special type. You swap a square numbered one through fifteen with the empty square (originally in the 16th position). This observation alone is not enough to prove that the 15 puzzle configuration suggested by Sam Loyd has no solution.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

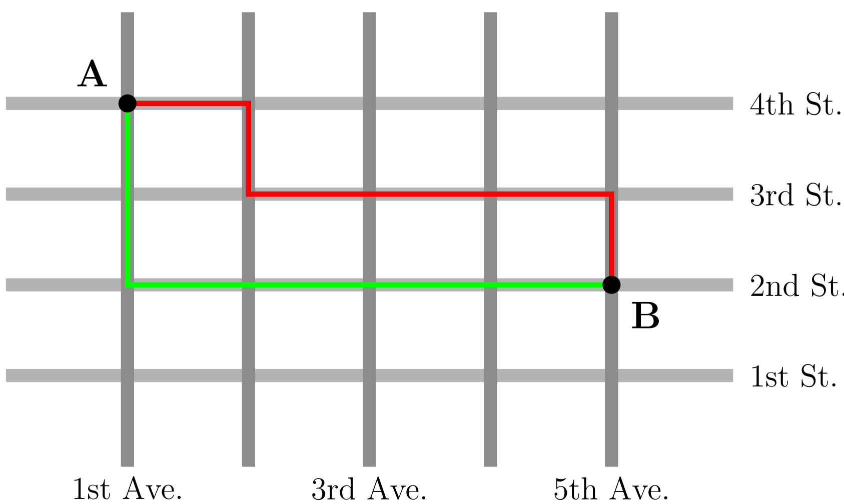
We need one more tool, called the taxicab geometry.

3 Taxicab Geometry

Imagine that you take a taxicab to get from point A to point B in a city with streets and avenues forming a rectangular pattern.



Similar to Euclidean geometry, there exists a shortest path. Unlike Euclidean geometry, the shortest path is not unique. For example, the green and red routes on the picture below are both shortest ways from A to B.



Problem 15 *On the picture above, draw a third shortest path from A to B.*

The point A lies at the intersection of the 1st Ave. and the 4th St. Let us write this fact down as follows.

$$A = (1, 4)$$

B lies at the intersection of the 5th Ave. and the 2nd St.

$$B = (5, 2)$$

Let a be the distance between two neighbouring avenues and let s be the distance between two neighbouring streets. No matter what shortest path the cab driver chooses, he needs to drive 4 blocks East and 2 blocks South.

$$d_{tc}(A, B) = 4a + 2s$$

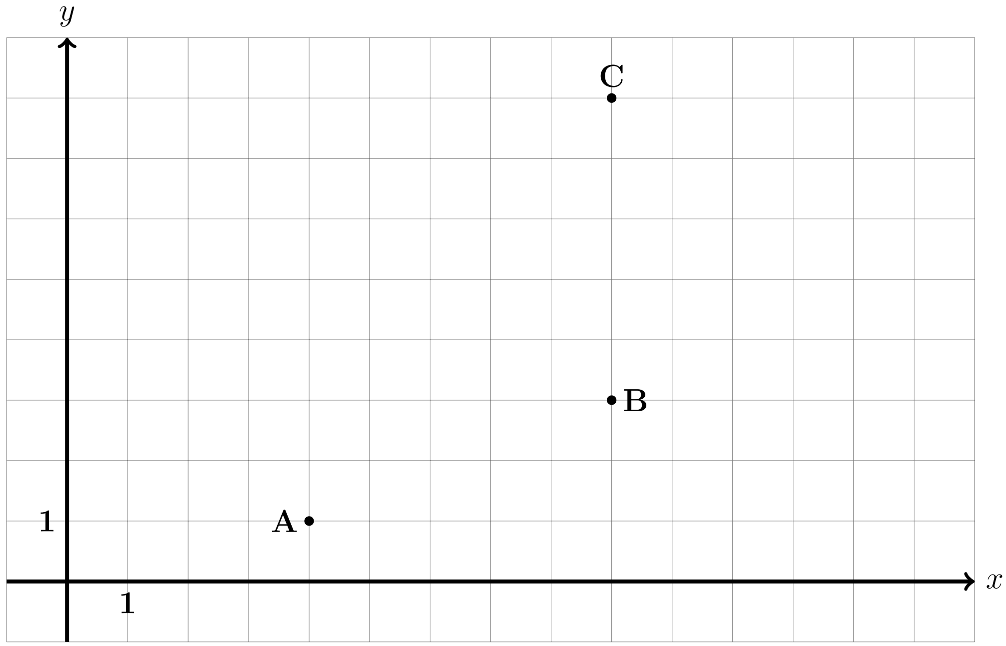
Problem 16 *Find the Euclidean distance $d_E(A, B)$ between the points A and B.*

$$d_E(A, B) =$$

Without doing any computations, put the correct sign, >, <, or =, between the distances below. Explain your choice.

$$d_E(A, B) \qquad\qquad d_{tc}(A, B)$$

Problem 17 *For the grid below, $a = s = 1$.*



Find the following taxicab distances.

$$d_{tc}(A, B) =$$

$$d_{tc}(A, C) =$$

$$d_{tc}(B, C) =$$

If we use the taxicab distance instead of the Euclidean one, would the triangle inequality hold for the triangle ABC?

For any two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in the coordinate plane, let us define the taxicab distance between them as follows.

$$d_{tc}(A, B) = |x_1 - x_2| + |y_1 - y_2|$$

Problem 18 *Find the taxicab distance between the points* $A = (-2, 7)$ *and* $B = (3, -5)$.

$$d_{tc}(A, B) =$$

Problem 19 *The taxicab distance between the points* A *and* B *is zero.*

$$d_{tc}(A, B) = 0$$

Can the points be different? Why or why not?

Note that the taxicab distance shares some basic properties with the Euclidean one. The distance from A to B equals the distance from B to A.

$$d_{tc}(A, B) = d_{tc}(B, A) \qquad\qquad d_E(A, B) = d_E(B, A)$$

In both cases, the distance between two points is zero if and only if the points coincide.

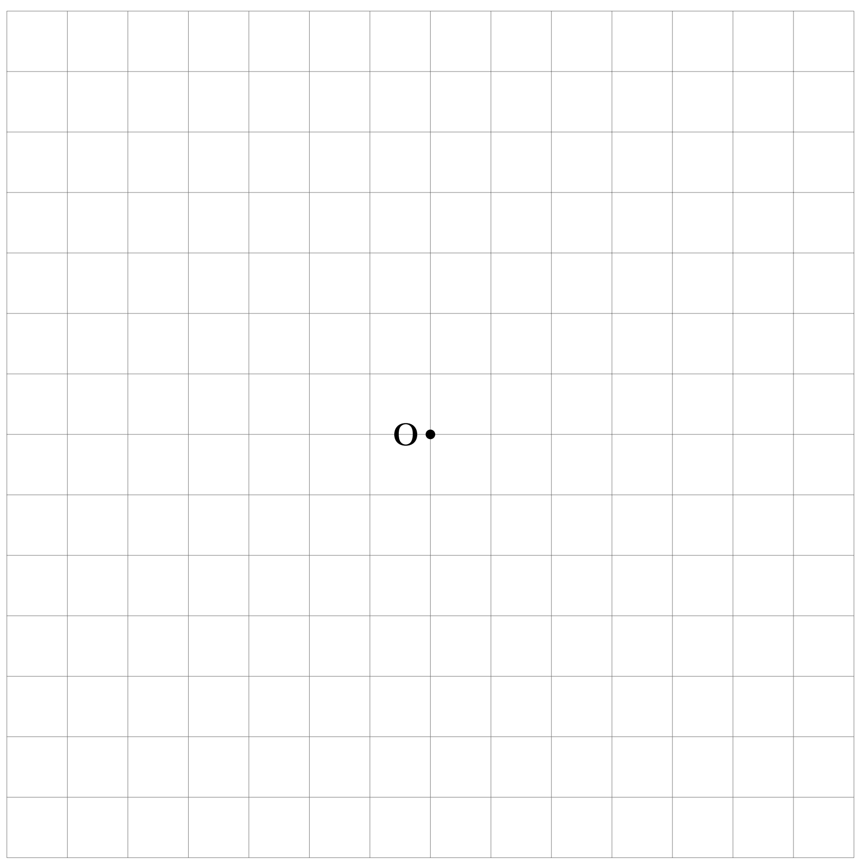
$$d_{tc}(A, B) = 0 \;\Leftrightarrow\; A = B \;\Leftrightarrow\; d_E(A, B) = 0$$

The distance between two different points is always positive.

$$A \neq B \;\Rightarrow\; d_{tc}(A, B) > 0 \;\text{ and }\; d_E(A, B) > 0$$

We have observed one difference between the distances in Problem 17. There, $d_{tc}(A, C) = d_{tc}(A, B) + d_{tc}(B, C)$. For the Euclidean distance, this means that the point B lies on the straight line AC between the points A and C, quite obviously not necessarily the case for the taxicab distance.

Problem 20 *On the grid below, mark all the points that have the taxicab distance 6 from the point O.*



Problem 21 *Give the definition of a circle of radius R centered at the point O in the space below.*

Is the figure you constructed in the previous problem a circle? Why or why not?

Problem 22 *Find the taxicab distance from the current position of the empty square to the lower-right corner of the 15 puzzle.*

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

4 Solving Loyd's Puzzle

Finally, we have all the tools we need to prove that Sam Loyd's configuration is unsolvable.

Let $\mathcal{P}$ be a function that assigns each configuration $\mathcal{C}$ of the 15 puzzle one of the two values, either zero or one. Let us set $\mathcal{P}(\mathcal{C}) = 0$ if the the sum of the inversions of $\mathcal{C}$ plus the taxicab distance from its empty square position to the lower-right corner of the puzzle is an even number. Let us set $\mathcal{P}(\mathcal{C}) = 1$ otherwise. For example, let us take another look at the configuration $\mathcal{C}$ we have considered in Problems 8 and 22.

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

The number of inversions of this configuration is 16. The taxicab distance from the empty square of the configuration to the lower-right corner is 3. The sum, $16+3 = 19$, is an odd number, so $\mathcal{P}(\mathcal{C}) = 1$.

Let us call a configuration $\mathcal{C}$ of the 15 puzzle *even* if $\mathcal{P}(\mathcal{C}) = 0$ and let us call it *odd* otherwise. This way, all the configurations of the puzzle are split into two classes, even and odd.

Problem 23 *Is the following configuration even or odd? Try to answer this question without doing too many calculations. Hint: compare this configuration to that on page 17.*

1	2	3	4
5		6	8
9	10	7	11
13	15	14	12

It's time for the big question! Why isn't Sam Loyd's puzzle solvable?

Problem 24 *Is the configuration in Sam Loyd's puzzle even or odd?*

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Theorem 2 *Odd configurations of the 15 puzzle are not solvable.*

Problem 25 *Prove Theorem 2 to show that Sam Loyd's puzzle is not solvable.*

Theorem 3 *Any even configuration of the 15 puzzle is solvable.*

Theorem 3 is not hard to prove using mathematical induction. We are not going to do it at the moment. The following theorem is much harder to prove.

Theorem 4 *Lengths of the optimal solutions of the 15 puzzle range from 0 to 80 single-tile moves.*