Nonstandard analysis

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Summer 2023

The conduit is available at `https://tinyurl.com/ORMCconduit`.

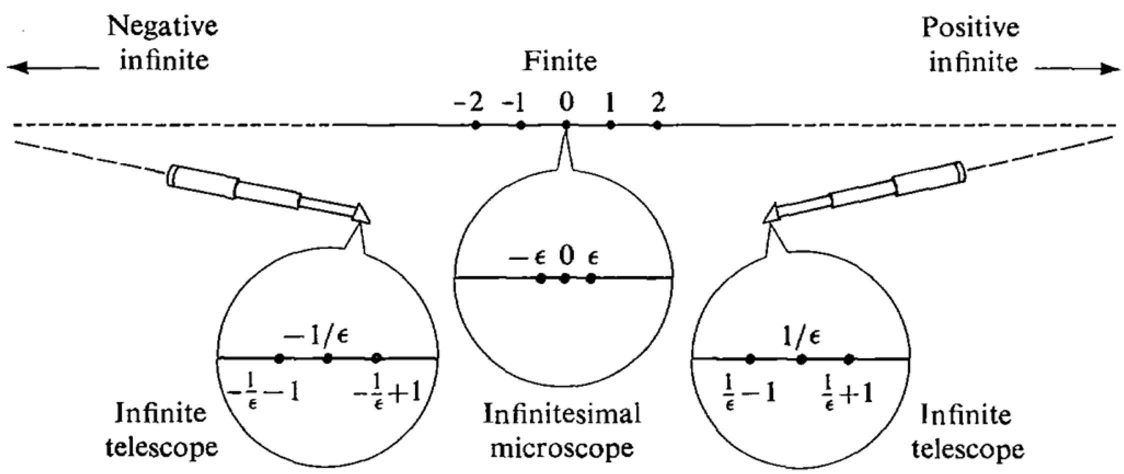
Problem 1 (✐). Let a be a hyperreal number such that $\operatorname{st}(a) = 3$, but $a \neq 3$. Find

$$\operatorname{st}\left(\frac{a^2 - 5a + 6}{a - 3}\right).$$

Infinitesimal Geometry

Basic definitions

Instead of using the terms "limited" and "unlimited", we may sometimes use "finite" and "infinite", although this can introduce some confusion.



Throughout this problem set, we will be working in a hyper-Euclidean plane ${}^*\mathbb{R}^2$, where points are denoted by (x, y). If P and Q are points, then PQ denotes a unique line passing through P and Q, provided P and Q are distinct. The distance between P and Q is denoted $|PQ|$.

Definition 1. Two points in the plane are considered *infinitesimally close* if the distance between them is infinitesimal. A point is called *limited*, if the distance from that point to some standard point is limited. A point is *nearstandard* if it is infinitesimally close to a standard point.

Problem 2 (✐). a) Prove that two points (x, y) and (x', y') are infinitesimally close if and only if $x \sim x'$ and $y \sim y'$.

b) Prove that a point (x, y) is limited if and only if x and y are limited.

c) Prove that a point is limited if and only if it is nearstandard and that any nearstandard point is infinitesimally close to a unique standard point, called its standard part.

Definition 2. Let F represent any nonstandard geometric figure. The *shadow* of F is then defined as the set of all standard points that are infinitesimally close to some point on F. It is denoted by $\text{sh}(F) = \{(x,y) \in \mathbb{R}^2 : \exists (x', y') \in F, \text{ such that } \text{st}(x', y') = (x,y)\}$

Definition 3. A line is *limited* if it passes through a limited point and nearstandard if its shadow is a line.

Theorem 1. *A line is limited if and only if it is nearstandard. Two limited lines intersecting at a limited point at an infinitesimal angle have the same shadow.*

Problem 3 (✐). Find a circle whose shadow is a line.

Curves and tangents

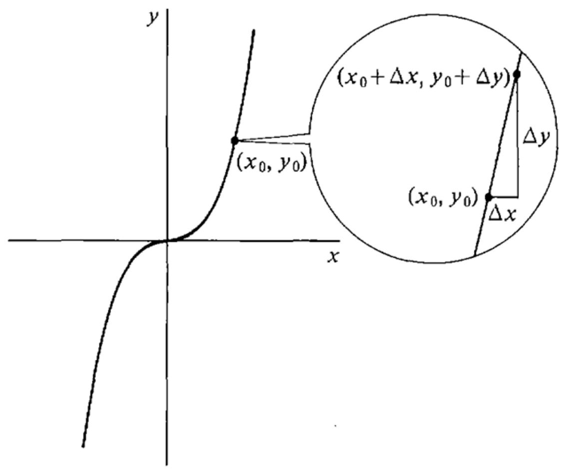
Definition 4. Let $C \subset \mathbb{R}^2$ be a plane curve. We say that a line l through a standard point P on C is a *tangent line* to C at P if for every point $Q \in {}^*C$ infinitesimally close to P, the line PQ is nearstandard and l is the shadow of PQ. A curve C is called *smooth* at P if there exists a tangent line to C at P. A curve C is *smooth* if it is smooth at any point.

Problem 4. Prove that a circle is a smooth curve and that the tangent to a circle is perpendicular to the radius.

Problem 5 (*). An ellipse is a set of points for which the sum of the distances to two given points (foci) is fixed. Prove that an ellipse is a smooth curve. Prove that any ray emanating from one focus of the ellipse and reflecting from it according to the laws of geometric optics (the angle of incidence equals the angle of reflection) will come to the other focus.

Problem 6. Let C be a smooth plane curve, and let Q be the point on C closest to P. Prove that PQ is normal to C, i.e. perpendicular to the tangent line to C at Q.

Problem 7 (✐). Write the equation of the tangent to the graph of the differentiable function $f(x)$ at the point x_0.



Problem 8. Prove that if $f(x_0)$ is the maximal or minimal value of the differentiable f, then $f'(x_0) = 0$, i.e. $\frac{f(x_0+")-f(x_0)}{"} \sim 0$ for all infinitesimals ε.

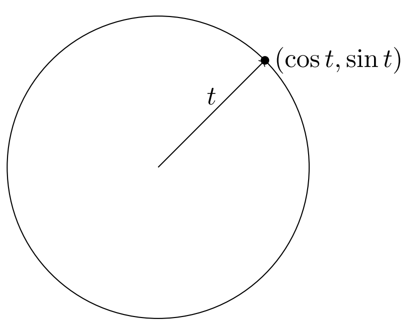
Problem 9 (✎). Given a point P inside an angle AOB, find a line through P that minimizes the area of the triangle it forms with the angle AOB.

Problem 10. Find the tangent to the parabola $y = \frac{x^2-3x+3}{3}$, which is parallel to the line $y = x$.

Problem 11 (✎). At what angle do the curves $y = x^2$ and $x = y^2$ intersect?

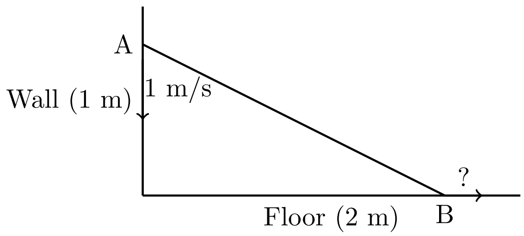
Trigonometric functions

Recall the following construction of the trigonometric functions: we allow a particle to start at (0,1) and move counterclockwise along the unit circle at unit speed. This means that the total length of the path traversed by the particle after time t is exactly t units. Let E_t be the position of the particle at time t. Its coordinates, are, by definition, called *cosine* $\cos t$ and *sine* $\sin t$, respectively.



Problem 12. Let t be a standard real number, and let ε be infinitesimal. Let $v = \overline{E_tE_{t+"}}$ be the vector of infinitesimal displacement of the particle. Prove that $\frac{|v|}{"} \sim 1$, and that $\angle OE_tE_{t+"} \sim 90°$. Find the standard parts of the components of the normalized vector $\frac{v}{"}$ from that information. These are exactly the derivatives of sine and cosine.

Problem 13 (✎). A stick AB is sliding down a wall. What is the speed of the point B?



Problem 14 (✎)**.** Consider the right triangle with legs 1 and t, so that the angle opposing t is exactly $\arctan t$. Extend the leg t to the length of $t + \varepsilon$ and find geometrically the infinitesimal change of angle. Find the derivative of $\arctan t$ from that. Do the same for $\arcsin t$.

Problem 15. Which of the following numbers are limited? Find the standard parts of those which are.

a) $\tan\left(\frac{}{2} - \varepsilon\right)$,

b) $\arctan(H)$,

c) $\frac{\sin(''^2)}{''\sin('')}$,

d*) $\frac{1-\cos('')}{''^2}$,

e*) $\frac{\sin('')-\tan('')}{''^3}$.

Problem 16 (Snell's Law ✎)**.** Let the speed of light in the air, v_1, and the speed of light in the water, v_2, be given. Assume that the light beam takes the fastest path from point A to point B, going from air to water. What is the relationship between the angle of incidence on the surface of water θ_1 and the angle of incidence under the surface of water θ_2?