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The 15 Puzzle

We will now begin to learn solving the *15 puzzle* (when a solution exists).

The puzzle consists of a 4×4 frame randomly filled with 15 squares numbered one through fifteen. The objective is to slide the squares in the proper order, left to right, starting with the top row as on the picture below.

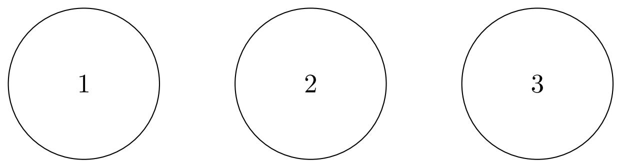
15 puzzle

You are encouraged to get the puzzle, either in the solid form or as a smartphone/tablet app, and to start playing!

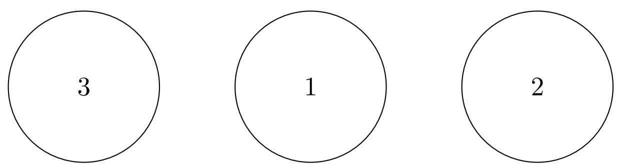
In the meanwhile, let's dive into permutations!

1 Permutations

Consider a set of marbles numbered 1 through n. Originally the marbles are lined up in the order given by their numbers. The following picture shows an example with $n = 3$.



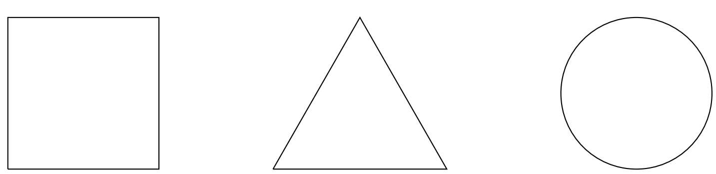
Then the marbles are reshuffled in a different order.



A *permutation* is the operation of reshuffling the marbles (or elements of any set). The one shown in the example is written down as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

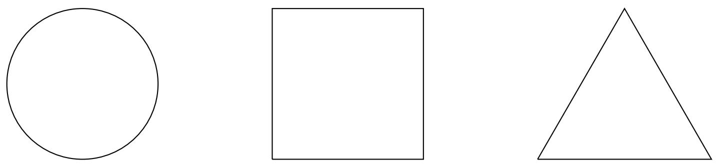
Instead of the numbered marbles, we can reshuffle distinguishable elements of any set. For example, let us consider the following geometric figures rather than the marbles numbered 1, 2, and 3.



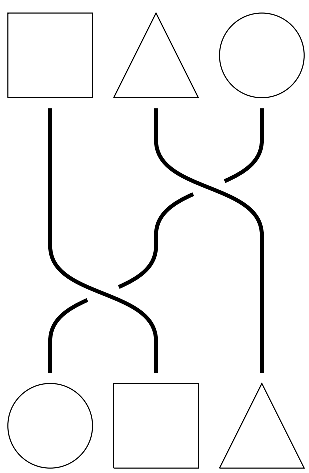
Then the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

will reshuffle the figures into the following order.

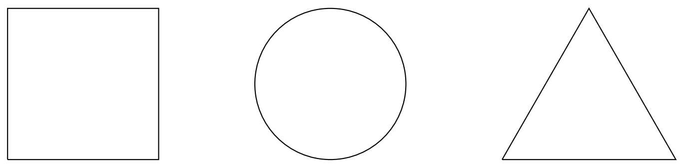


A more pictorial, geometric way is to think about a permutation as of a system of highways connecting a number of entry points to the equal number of exits. The highways do not intersect, they go either above or below one another transporting the objects from the original positions at the top to the new positions at the bottom.

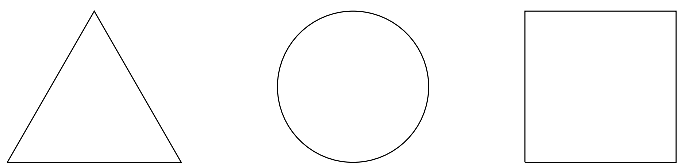


$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

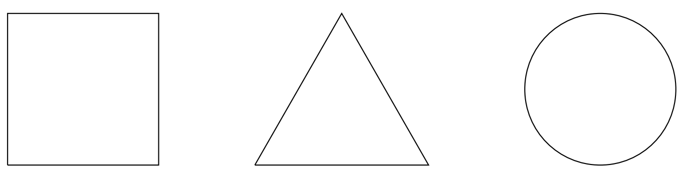
Problem 1.1 *For the following figures, write down the permutations that correspond to the following pictures.*



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that the last permutation does not reshuffle anything at all. Permutations of this kind typically denoted as e and called *trivial*. A trivial permutation is still a permutation, and an important one!

Problem 1.2 *Write down the trivial permutation for* $n = 5$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

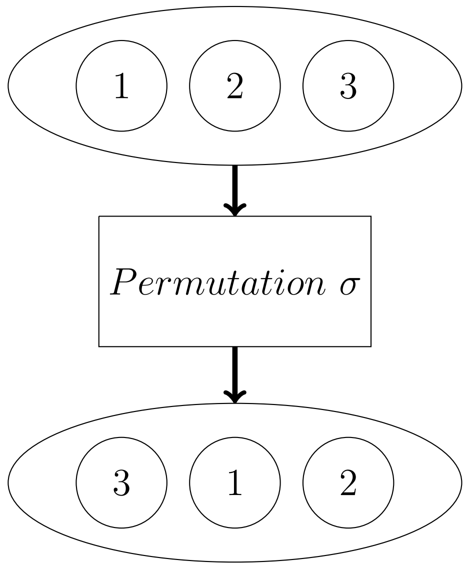
Problem 1.3 *For the original order of figures given on page 4, draw the figures in the orders prescribed by the permutations below. Use the space to the right of a permutation to draw the corresponding picture.*

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

We have described above a combinatorial and a geometric way to think of permutations. We will now introduce the third, functional, approach.

A permutation is a bijection from an ordered set of n objects to itself. In the case below, the input is the list at the top of the diagram: the marble marked 1 in the first position, the marble marked 2 in the second position, and the marble marked 3 in the third one. The permutation transforms this list into the list at the bottom: the marble marked 3 in the first position, the marble marked 1 in the second, and the marble marled 2 in the third.



$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

The functional approach allows us to write $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. Note that the columns of the permutation provide the table of values for the function.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Problem 1.4 *Find $\delta(1)$, $\delta(2)$, and $\delta(3)$ for the permutations in Problem 1.1.*

To generalize slightly, let us consider a permutation α as a bijective function acting on a set of n elements. Then the following is its table of values.

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n-1) & \alpha(n) \end{pmatrix}$$

We have shown you three different ways to describe permutations: combinatorial, geometrical, and functional. Choose the one that speaks towards the deepest, darkest pit in your soul, but bear in mind that they all essentially define the same thing.

2 Counting Permutations

Let n be a positive integer. The following product is denoted as $n!$ and is called n *factorial*.

$$n! = n \times (n-1) \times \ldots \times 2 \times 1$$

Problem 2.1 *Compute* $5!$*.*

Problem 2.2 *How many permutations of four elements are there?*

Problem 2.3 *How many permutations of* $n+1$ *elements are there?*

Problem 2.4 *Write down a permutation of four elements.*

Problem 2.5 *Write down a permutation of four elements that keeps the third element in place.*

Find the number of permutations of four elements that keep the third element in place.

3 Product of Permutations

It is possible to combine, or *multiply*, permutations. For example, let us apply the permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

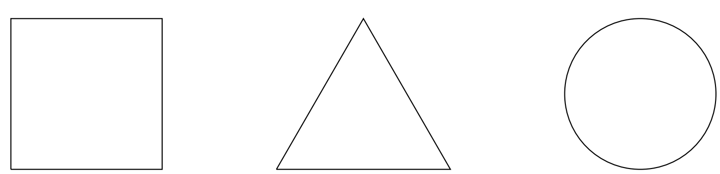
to the marbles already reshuffled by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The permutation δ switches the first and second elements, so

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

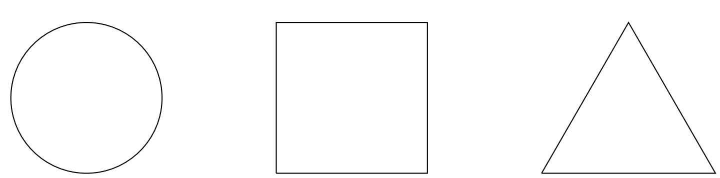
Let us take another look at the above computation using the figures from page 4. Originally, the set of the figures is ordered as follows.



The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

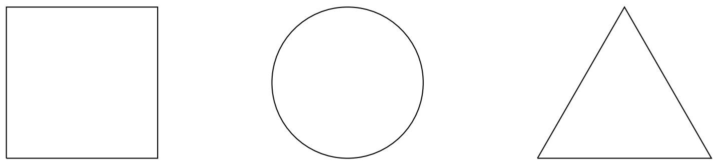
produces the picture:



The permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

applied to the latter configuration gives us the following.



Comparing the last picture to the original gives us the answer.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note that in the product $\delta \circ \sigma$ of permutations, it is the one on the right, σ, that acts first on the set it permutes!

We can also see how the two functions compose by writing them out like so:

$$\begin{aligned} 1 \mapsto 2 \mapsto 1 \\ 2 \mapsto 3 \mapsto 3 \\ 3 \mapsto 1 \mapsto 2 \end{aligned}$$

Problem 3.1 *Find the permutation $\sigma \circ \delta$.*

$$\sigma \circ \delta =$$

Problem 3.2 *Is the multiplication of permutations commutative?*

Problem 3.3 *Find two non-trivial permutations of four elements that do commute.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 3.4 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

If you need to use a pictorial representation as a tool, take the one on page 4 and add a diamond $\Diamond$ as the fourth figure.

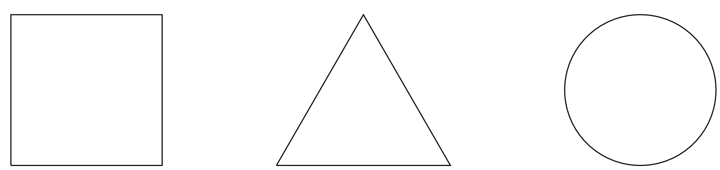
Problem 3.5 *Find the product* $\sigma \circ \delta$ *of the permutations from Problem 3.4. If needed, use a pictorial representation. Do the permutations* δ *and* σ *commute?*

4 Inverse Permutation

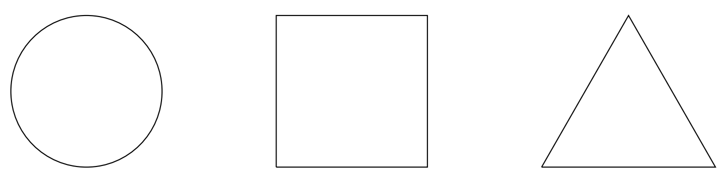
A permutation δ is called *opposite* to a permutation σ if $\delta \circ \sigma = e$. In other words, δ undoes what σ does. Such a permutation is denoted as σ^{-1} and is called the *permutation opposite to sigma* or *sigma inverse*.

Example 4.1 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

The permutation σ *reshuffles the figures*



in the following order.



Hence, $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

Note that since the permutation σ^{-1} undoes what the permutation σ does, σ works the same way for σ^{-1}. Hence, not only $\sigma^{-1} \circ \sigma = e$, but $\sigma \circ \sigma^{-1} = e$ as well. Thus, σ and σ^{-1} always commute.

$$\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e$$

Problem 4.1 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Problem 4.2 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.

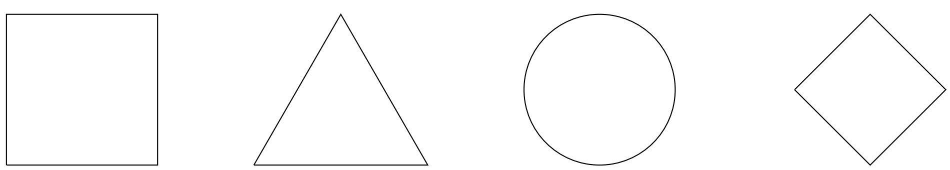
Problems 4.1 and 4.2 exhibit two different non-trivial permutations σ that are self-inverse, $\sigma^{-1} = \sigma$. Similarly, in regards to numbers, there exists only two such values where $x^{-1} = x$, 1 and -1. Unlike numbers, however, there exist lots of different non-trivial self-inverse permutations.

Problem 4.3 *Find a non-trivial permutation σ different from the ones in Problems 4.1 and 4.2 such that $\sigma^{-1} = \sigma$.*

Problem 4.4 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

If you cannot do it right away, please use the following pictorial representation for the original arrangement.



5 Why are we here?

So what's up with this fandangled permutation business anyways? What's its purpose and how is it related to the 15-puzzle?

Well, I don't really know. I know it's got something to do with math though. But, an unnamed source says that we can solve a popular problem regrading the 15 puzzle as the mathematical foundation of the solution is the theory of permutations. The theory not only helps to unravel the puzzle, but also comes quite handy in a wide variety of applications.

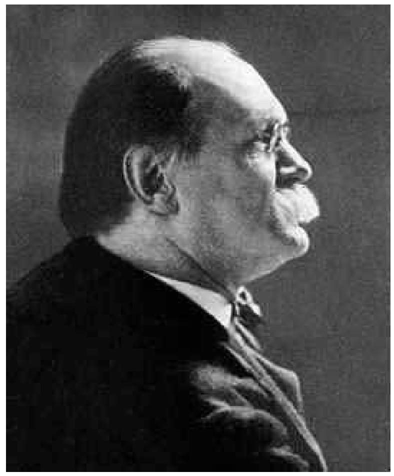
The 15 puzzle was invented by Noyes Palmer Chapman, a postmaster in Canastota, New York, in the mid-1870s. Sam Loyd, a man of interesting character [1] and prominent American chess player,[2] has offered $1,000 (about $25,000 of modern day money) for solving the puzzle in the form shown on the picture below.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

[1] Loyd was simultaneously called "America's greatest puzzler," "hustler," and "fast talking snake-oil salesman" by contemporary sources.

[2] Ranked 15th in the world.

As it turns out, Loyd's puzzle permutation is unsolvable, and finding out why is the purpose of this mini-course.



Sam Loyd, 1841 – 1911

Problem 5.1 *Write down the permutation corresponding to the Loyd's puzzle.*

Problem 5.2 *Let us call σ the permutation from Problem 5.1. Find σ^{-1}.*

6 Improving our Notation

Note that the first line of the notation we have used for writing down permutations so far is redundant. Indeed

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

means that we shuffle the second element to the first position, the first element to the second position, the fourth element to the third position, and the third element to the fourth one. Without any loss of clarity, we can write this down as

$$\sigma = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}.$$

Problem 6.1 *Apply the permutation* $\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$ *to the sequence of geometric figures on page 15 and draw the result in the space below.*

Problem 6.2 *Find the product* $\delta \circ \sigma$ *of the following two permutations.*

$$\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} \qquad \delta = \begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix}$$

7 Cycles

Let us set $\sigma^0 = e$ for any permutation σ. The permutation σ^2 is defines as $\sigma \circ \sigma$, σ^3 as $\sigma \circ \sigma^2$, and so on. Similarly, $\sigma^{-2} = \sigma^{-1} \circ \sigma^{-1}$, $\sigma^{-3} = \sigma^{-1} \circ \sigma^{-2}$, and so forth.

Problem 7.1 *Find the following powers of the permutation* $\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$.

$$\sigma^2 =$$

$$\sigma^3 =$$

$$\sigma^4 =$$

$$\sigma^{-1} =$$

$$\sigma^{-2} =$$

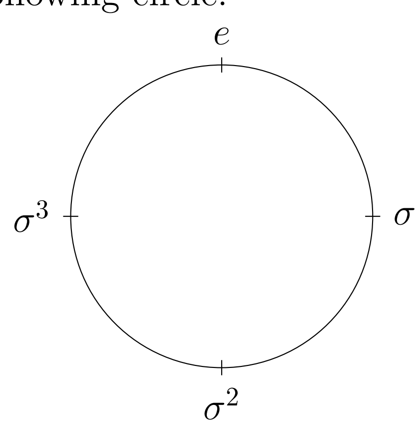
$$\sigma^{-3} =$$

$$\sigma^{-4} =$$

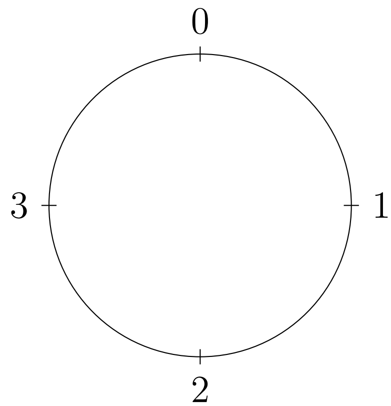
Solving Problem 7.1, you may have noticed the following. The formula $\sigma^4 = e$ means that

- $\sigma \circ \sigma^3 = e$, hence $\sigma^{-1} = \sigma^3$ and $\sigma^{-3} = \sigma$;
- $\sigma^2 \circ \sigma^2 = e$, hence $\sigma^{-2} = \sigma^2$. Furthermore,
- $\sigma^5 = \sigma^4 \circ \sigma = e \circ \sigma = \sigma$;
- $\sigma^6 = \sigma^4 \circ \sigma^2 = e \circ \sigma^2 = \sigma^2$;
- $\sigma^7 = \sigma^4 \circ \sigma^3 = e \circ \sigma^3 = \sigma^3$;
- $\sigma^8 = \sigma^4 \circ \sigma^4 = e \circ e = e$;
- $\sigma^9 = \sigma^8 \circ \sigma = e \circ \sigma = \sigma$;
- $\sigma^{-5} = \sigma^{-4} \circ \sigma^{-1} = e^{-1} \circ \sigma^3 = e \circ \sigma^3 = \sigma^3$; and so forth.

It turns out that all the powers of the permutation σ reside naturally on the following circle.



To understand the circle, consider the integers on a circle divided into four equal parts.



On the circle, 0 coincides with 4. We write this fact down as

$$4 \equiv 0 \ (mod\ 4)$$

and read it as *4 is congruent to 0 modulo 4*. The usual "=" sign is reserved for the straight number line; we use "≡" on the circle instead. The *mod* 4 symbol tells us that the circle is divided into 4 equal parts, so 4 coincides with 0, 5 with 1, 6 with 2, and so on. Or in the new notations, $4 \equiv 0 \ (mod\ 4)$, $5 \equiv 1 \ (mod\ 4)$, $6 \equiv 2 \ (mod\ 4)$, $7 \equiv 3 \ (mod\ 4)$, and so forth.

Problem 7.2

$$-21 \equiv \qquad\qquad (mod\ 4)$$

$$6 + 5 \equiv \qquad\qquad (mod\ 4)$$

As we can see, powers of the permutation σ from Problems 6.2 and 7.1 produce nothing more than a multiplicative realization of the *mod* 4 arithmetic. In other words, the *mod* 4 integers on the second circle serve as powers of the permutation σ on the first circle.

Example 7.1 *Find the 125 power of the permutation σ from Problems 6.2 and 7.1.*

$$\sigma^{125} = \sigma^{1 \ (mod \ 4)} = \sigma$$

Problem 7.3 *Find the -333 power of the permutation σ from Problems 6.2 and 7.1.*

The smallest positive power n of a permutation δ such that $\delta^n = e$ is called the *order of the permutation.*

Problem 7.4 *What is the order of the permutation σ we have considered in Problems 6.2, 7.1, and 7.3?*

Problem 7.5 *Find the following powers of the permutation* $\mu = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$.

$$\mu^2 =$$

$$\mu^3 =$$

$$\mu^4 =$$

$$\mu^{-1} =$$

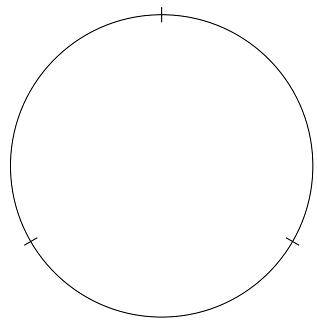
$$\mu^{-2} =$$

$$\mu^{-3} =$$

What is the order of the permutation μ*?*

The problem continues on the next page.

Mark μ^{123}, μ^{124}, and μ^{125} on the circle below.



What mod n arithmetic is realized by the powers of μ?

8 Improving our Notation, Again... :)

Let us take another look at the permutation μ from Problem 7.5.

$$\mu = \left(3 \quad 2 \quad 4 \quad 1 \right)$$

The permutation does not shuffle the second element. Hence, writing it is redundant. Knowing that the original set consists of four elements, we can write the permutation down as

$$\mu = \left(3 \quad 4 \quad 1 \right)$$

Since the second element does not appear in the formula, we know that the permutation does not move it. This convention becomes very convenient with larger permutations. For example, let us take another look at Sam Loyd's formulation of the

15 puzzle. Since we need to keep track of the empty square, as well as of the numbered ones, let us consider it as the 16th tile.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix}$$

switches the 14th and 15th elements only. Writing down the 14 elements it does not move is a waste of time! In the new notations,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix} = \begin{pmatrix} 15 & 14 \end{pmatrix}.$$

Since all other elements are not mentioned, we know that the permutation does not shuffle them.

Here is one more example. Let $\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$ be a permutation of six elements. Since the elements 2, 5, and 6 are not listed, μ keeps them in place. So in fact, $\mu = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix}$.

Problem 8.1 *The permutation* $\nu = \begin{pmatrix} 3 & 1 \end{pmatrix}$ *acts on a set of three elements. Write down its full version.*

$$\nu =$$

What is the order of ν*?*

Write down the short form of $\nu^{-10,000,831}$.

$$\nu^{-10,000,831} =$$

Problem 8.2 *The permutation* $\delta = \begin{pmatrix} 3 & 5 & 7 & 1 \end{pmatrix}$ *acts on a set of seven elements. Write down its full version.*

$$\delta =$$

What is the order of δ*?*

Write down the short form of $\delta^{-10,000,000}$.

$$\delta^{-10,000,000} =$$

9 Transposition

A permutation that swaps two elements and doesn't shuffle anything else is called a *transposition*. For example, the permutation that switches the order of the third and fifth element in a six-element set is $\left(\begin{array}{ll} 5 & 3 \end{array}\right) = \left(\begin{array}{llllll} 1 & 2 & 5 & 4 & 3 & 6 \end{array}\right)$.

Problem 9.1 *What is the inverse of the transposition* $\left(\begin{array}{ll} 5 & 3 \end{array}\right)$?

$$\left(5\ 3\right)^{-1} =$$

Problem 9.2 *What is the order of any transposition?*

Any permutation can be realized as a product of transpositions. For example, let us consider the permutation $\sigma = \left(\begin{array}{llll} 3 & 1 & 4 & 2 \end{array}\right)$ from Problems 6.2, 7.1, and 7.3. Applying the transposition $\left(\begin{array}{ll} 2 & 1 \end{array}\right)$ to the original order of the elements gives us the following.

$$\left(\begin{array}{llll} 1 & 2 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{llll} 2 & 1 & 3 & 4 \end{array}\right)$$

Let us apply the transposition $\left(\begin{array}{ll} 4 & 1 \end{array}\right)$ to the result.

$$\left(\begin{array}{llll} 2 & 1 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{llll} 4 & 1 & 3 & 2 \end{array}\right)$$

Finally, applying the transposition $\left(\begin{array}{ll} 3 & 1 \end{array}\right)$ finishes the job.

$$\left(\begin{array}{llll} 4 & 1 & 3 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{llll} 3 & 1 & 4 & 2 \end{array}\right)$$

Or more concisely,

$$\left(\begin{array}{llll} 3 & 1 & 4 & 2 \end{array}\right) = \left(\begin{array}{ll} 3 & 1 \end{array}\right)\left(\begin{array}{ll} 4 & 1 \end{array}\right)\left(\begin{array}{ll} 2 & 1 \end{array}\right).$$

Problem 9.3 *Realize the permutation* $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$ *as a product of transpositions.*

Problem 9.4 *Write down the permutation* μ *that corresponds to the following move of the 15 puzzle. Remember, we treat the empty square as the 16th tile!*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

$\longrightarrow$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	
13	**15**	**14**	**12**

$$\mu =$$

Find the product $\mu \circ \begin{pmatrix} 15 & 14 \end{pmatrix}$ *and compare the answer to the order of the squares on the second picture of the previous page.*

$$\mu \circ \begin{pmatrix} 15 & 14 \end{pmatrix} =$$

Problem 9.5 *Write down the permutation that corresponds to the following move of the 15 puzzle. Remember, we treat the empty square as the 16th tile!*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	
13	**15**	**14**	**12**

$\longrightarrow$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**		**11**
13	**15**	**14**	**12**

If you are finished doing all the above, but there still remains some time...

Recall that cryptarithms, also known as alphametics, are math games of figuring out unknown numbers represented by words. Different letters correspond to different digits. The first digit of a number cannot be zero.

Problem 9.6 *Solve the following cryptarithm.*

$$\begin{array}{r} N\ U\ M\ B\ E\ R \\ +\ N\ U\ M\ B\ E\ R \\ \hline P\ U\ Z\ Z\ L\ E \end{array}$$