Nonstandard analysis

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The conduit is available at `https://tinyurl.com/ORMCconduit`.

Analysis of Infinitesimals

Suppose we have two quantities x and y (say, the radius of a circle and its area), linked by a functional relationship: knowing one, we can calculate the other. Let's change them slightly $x \to x + \Delta x$, $y \to y + \Delta y$, preserving the relationship. The ratio $\frac{\Delta y}{\Delta x}$ (more precisely, its limit when the changes are infinitesimally small) is called the derivative of y with respect to x and is denoted $\frac{dy}{dx}$. If the relationship is given by a function f, that is, $y = f(x)$, then we talk about the derivative of the function f at the point x and denote this derivative $f'(x)$:

$$f'(x) = \operatorname{st}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right).$$

Problem 1. Using this definition, calculate the derivatives of the following functions:

a) $f(x) = x^2$

b) $f(x) = (x+3)^2$

c) $f(x) = x^2 + 3$

d) $f(x) = \frac{1}{x}$

e) $f(x) = \sqrt{x}$

Problem 2. Fill in the blanks, following the example: The speed of a point on a line is the derivative of its ⟨coordinate⟩ with respect to ⟨time⟩.

a) The acceleration of a point on a line is the derivative of ⟨...⟩ with respect to ⟨...⟩.

b) The heat capacity of a body is the derivative of its internal ⟨...⟩ with respect to ⟨...⟩.

c) The current through a capacitor is the derivative of its ⟨...⟩ with respect to ⟨...⟩.

d) The circumference of a circle is the derivative of ⟨...⟩ of this circle with respect to ⟨...⟩.

e) The area of the surface of a sphere is the derivative of its ⟨...⟩ with respect to ⟨...⟩.

f) The slope of the graph is the derivative of ⟨...⟩ with respect to ⟨...⟩.

Problem 3. The width of a rectangle is 2 m and grows at a rate of 1 mm/s. The height of the rectangle is 1 m and grows at a rate of 3 mm/s. What are the rates at which the perimeter and area of this rectangle are growing?

Problem 4. A point X moves along a unit circle at a unit speed: at time t its coordinates are $(\cos t, \sin t)$. In what direction is its velocity, and what is its value (i.e., what are its coordinates)?

Problem 5. A stick AB is sliding down a wall. What is the speed of the point B?