Definable Sets

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Part 1: Logical Algebra

Definition 1:
Logical operators operate on the values $\{\text{True}, \text{False}\}$,
just like algebraic operators operate on numbers.
In this handout, we'll use the following operators:
- $\neg$: not
- $\wedge$: and
- $\vee$: or
- $\rightarrow$: implies
- $()$, parenthesis.

The function of these is defined by *truth tables*:

and

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

or

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

implies

A	B	$A \rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

not

A	$\neg A$
T	F
F	T

$A \wedge B$ is only true if both A and B are true. $A \vee B$ is true when A or B (or both) are true.
$\neg A$ is the opposite of A, which is why it looks like a "negative" sign.

$A \rightarrow B$ is a bit harder to understand. Read aloud, this is "A implies B."
The only time $\rightarrow$ is false is when $T \rightarrow F$. This may seem counterintuitive, but it will make more sense as we progress through this handout.

Problem 2:
Evaluate the following.
- $\neg T$
- $F \vee T$
- $T \wedge T$
- $(T \wedge F) \vee T$
- $(T \wedge F) \vee T$
- $(\neg(F \vee \neg T)) \rightarrow T$
- $(F \rightarrow T) \rightarrow (\neg F \vee \neg T)$

Problem 3:
Evaluate the following.

- $A \to T$ for any A
- $(\neg(A \to B)) \to A$ for any A, B
- $(A \to B) \to (\neg B \to \neg A)$ for any A, B

Problem 4:
Show that $\neg(A \to \neg B)$ is equivalent to $A \land B$.
That is, show that these give the same result for the same A and B.
Hint: Use a truth table

Problem 5:
Can you express $A \lor B$ using only $\neg$, $\to$, and $()$?

Note that both $\land$ and $\lor$ can be defined using the other logical symbols.
The only logical symbols we *need* are $\neg$, $\to$, and $()$.
We include $\land$ and $\lor$ to simplify our logical expressions.

Part 2: Structures

Definition 6:
A *universe* is a set of meaningless objects. Here are a few examples:
- $\{a, b, ..., z\}$
- $\{0, 1\}$
- $\mathbb{Z}$, $\mathbb{R}$, etc.

Definition 7:
A *structure* consists of a universe U and a set of symbols.
A structure's symbols give meaning to the objects in its universe.

Symbols come in three types:
- Constant symbols, which let us specify specific elements of our universe.
  Examples: $0, 1, \frac{1}{2}, \pi$
- Function symbols, which let us navigate between elements of our universe.
  Examples: $+, \times, \sin x, \sqrt{x}$
- Relation symbols, which let us compare elements of our universe.
  Examples: $<, >, \leq, \geq$

The equality check $=$ is **not** a relation symbol. It is included in every structure by default.

Example 8:
The first structure we'll look at is the following:

$$\Big(\mathbb{Z} \mid \{0, 1, +, -, <\}\Big)$$

This is a structure with the universe $\mathbb{Z}$ that contains the following symbols:
- Constants: $\{0, 1\}$
- Functions: $\{+, -\}$
- Relations: $\{<\}$

If you look at our set of constant symbols, you'll see that the only integers we can directly refer to in this structure are 0 and 1. If we want any others, we must define them using the tools this structure offers.

Say we want the number 2. We could use the function $+$ to define it: $2 \coloneqq [x \text{ where } 1 + 1 = x]$
We would write this as $2 \coloneqq [x \text{ where } +(1, 1) = x]$ in proper "functional" notation.

Problem 9:
Can we define -1 in $\Big(\mathbb{Z} \mid \{0, 1, +, -, <\}\Big)$? If so, how?

Problem 10:
Can we define -1 in $\Big(\mathbb{Z} \mid \{0, +, -, <\}\Big)$?
Hint: In this problem, 1 has been removed from the set of constant symbols.

Let us formalize what we found in the previous two problems.

Definition 11:
A *formula* in a structure S is a well-formed string of constants, functions, and relations.

You already know what a "well-formed" string is: $1 + 1$ is fine, $\sqrt{+}$ is nonsense.
For the sake of time, I will not provide a formal definition. It isn't particularly interesting.

A formula can contain one or more *free variables.* These are denoted $\varphi(a, b, ...)$.
Formulas with free variables let us define "properties" that certain objects have.
For example, x is a free variable in the formula $\varphi(x) = [x > 0]$.
$\varphi(3)$ is true and $\varphi(-3)$ is false.

Definition 12: Definable Elements
Say S is a structure with a universe U.
We say an element $e \in U$ is *definable in S* if we can write a formula that only e satisfies.

Problem 13:
Can we define 2 in the structure $\left(\mathbb{Z}^+ \mid \{4, \times\}\right)$?
Hint: $\mathbb{Z}^+ = \{1, 2, 3, ...\}$. Also, $2 \times 2 = 4$.

Problem 14:
Try to define 2 in the structure $\left(\mathbb{Z} \mid \{4, \times\}\right)$.

Problem 15:
What numbers are definable in the structure $\left(\mathbb{R}^+_0 \mid \{1, 2, \div\}\right)$?

Part 3: Quantifiers

Recall the logical symbols we introduced earlier: $(), \wedge, \vee, \neg, \rightarrow$
We will now add two more: $\forall$ (for all) and $\exists$ (exists).

Definition 16:
$\forall$ and $\exists$ are *quantifiers*. They allow us to make statements about arbitrary symbols.

Let's look at $\forall$ first. Let $\varphi(x)$ be a formula.
Then, the formula $\forall x \; \varphi(x)$ says "φ is true for all possible x."

For example, take the formula $\forall x \; (0 < x)$.
In English, this means "For any x, x is bigger than zero," or simply "Any x is positive."

$\exists$ is very similar: the formula $\exists x \; \varphi(x)$ states that there is at least one x that makes φ true.
For example, $\exists \; (0 < x)$ means "there is a positive number in our set".

Problem 17:
Which of the following are true in $\mathbb{Z}$?
Which are true in $\mathbb{R}^+_0$?
Hint: $\mathbb{R}^+_0$ is the set of positive real numbers and zero.

- $\forall x \; (x \geq 0)$
- $\neg(\exists x \; (x = 0))$
- $\forall x \; [\exists y \; (y \times y = x)]$
- $\forall xy \; \exists z \; (x < z < y)$ This is a compact way to write $\forall x \; (\forall y \; (\exists z \; (x < z < y)))$
- $\neg \exists x \; (\forall y \; (x < y))$

Problem 18:
Does the order of $\forall$ and $\exists$ in a formula matter?
What's the difference between $\exists x \; \forall y \; (x < y)$ and $\forall y \; \exists x \; (x < y)$?
Hint: In $\mathbb{R}^+$, the first is false and the second is true. $\mathbb{R}^+$ does not contain zero.

Problem 19:
Define 0 in $\left(\mathbb{Z} \mid \{\times\}\right)$

Problem 20:
Define 1 in $\left(\mathbb{Z} \mid \{\times\}\right)$

Problem 21:
Define -1 in $\left(\mathbb{Z} \mid \{0,<\}\right)$

Problem 22:
Let $\varphi(x)$ be a formula.
Define $(\forall x\ \varphi(x))$ using logical symbols and $\exists$.

Part 4: Definable Sets

Armed with $(), \wedge, \vee, \neg, \rightarrow, \forall$, and $\exists$, we have enough tools to define sets.

Definition 23: Set-Builder Notation
Say we have a condition c.
The set of all elements that satisfy that condition can be written as follows:

$$\{x \mid c \text{ is true}\}$$

This is read "The set of x where c is true" or "The set of x that satisfy c."

For example, take the formula $\varphi(x) = \exists y \ (y + y = x)$.
The set of all even integers can then be written

$$\{x \mid \varphi(x)\} = \{x \mid \exists y \ (y + y = x)\}$$

Definition 24: Definable Sets
Let S be a structure with a universe U.
We say a subset M of U is *definable* if we can write a formula that is true for some x iff $x \in M$.

For example, consider the structure $\Big(\mathbb{Z} \mid \{+\}\Big)$

Only even numbers satisfy the formula $\varphi(x) = \exists y \ (y + y = x)$,
So we can define "the set of even numbers" as $\{x \mid \exists y \ (y + y = x)\}$.
Remember—we can only use symbols that are available in our structure!

Problem 25:
Is the empty set definable in any structure?

Problem 26:
Define $\{0, 1\}$ in $\Big(\mathbb{Z}_0^+ \mid \{<\}\Big)$

Problem 27:
Define the set of prime numbers in $\Big(\mathbb{Z} \mid \{\times, \div, <\}\Big)$

Problem 28:
Define the set of nonreal numbers in $\Big(\mathbb{C} \mid \{\mathrm{real}(z)\}\Big)$
Hint: real(z) gives the real part of a complex number: $\mathrm{real}(3 + 2i) = 3$
Hint: z is nonreal if $x \in \mathbb{C}$ and $x \notin \mathbb{R}$

Problem 29:
Define $\mathbb{R}_0^+$ in $\Big(\mathbb{R} \mid \{\times\}\Big)$

Problem 30:
Let $\triangle$ be a relational symbol. $a \triangle b$ holds iff a divides b.
Define the set of prime numbers in $\Big(\mathbb{Z}^+ \mid \{\triangle\}\Big)$

Theorem 31: Lagrange's Four Square Theorem
Every natural number may be written as a sum of four integer squares.

Problem 32:
Define $\mathbb{Z}_0^+$ in $\Big(\mathbb{Z} \mid \{\times, +\}\Big)$

Problem 33:
Define $<$ in $\Big(\mathbb{Z} \mid \{\times, +\}\Big)$
Hint: We can't formally define a relation yet. Don't worry about that for now.
You can repharase this question as "given $a, b \in \mathbb{Z}$, can you write a sentence that is true iff $a < b$?"

Problem 34:
Consider the structure $S = (\mathbb{R} \mid \{0, \diamond\})$
The relation $a \diamond b$ holds if $|a - b| = 1$

Part 1:
Define 0 in S.

Part 2:
Define $\{-1, 1\}$ in S.

Part 3:
Define $\{-2, 2\}$ in S.

Problem 35:
Let P be the set of all subsets of $\mathbb{Z}_0^+$. This is called a *power set*.
Let S be the stucture $(P \mid \{\subseteq\})$

Part 1:
Show that the empty set is definable in S.
Hint: Defining $\{\}$ with $\{x \mid x \neq x\}$ is **not** what we need here.
We need $\varnothing \in P$, the "empty set" element in the power set of $\mathbb{Z}_0^+$.

Part 2:
Let $x \circledast y$ be a relation on P. $x \circledast y$ holds if $x \cap y \neq \{\}$.
Show that $\circledast$ is definable in S.

Part 3:
Let f be a function on P defined by $f(x) = \mathbb{Z}_0^+ - x$. This is called the *complement* of the set x.
Show that f is definable in S.

Part 5: Equivalence

Notation:
Let S be a structure and φ a formula.
If φ is true in S, we write $S \models \varphi$.
This is read "S satisfies φ"

Definition 36:
Let S and T be structures.
We say S and T are *equivalent* and write $S \equiv T$ if for any formula φ, $S \models \varphi \Longleftrightarrow T \models \varphi$.
If S and T are not equivalent, we write $S \not\equiv T$.

Problem 37:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{R} \mid \{+, 0\}\Big)$

Problem 38:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{N} \mid \{+, 0\}\Big)$

Problem 39:
Show that $\Big(\mathbb{R} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{N} \mid \{+, 0\}\Big)$

Problem 40:
Show that $\Big(\mathbb{R} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{Z}^2 \mid \{+, 0\}\Big)$

Problem 41:
Show that $\Big(\mathbb{Z} \mid \{+, 0\}\Big) \not\equiv \Big(\mathbb{Z}^2 \mid \{+, 0\}\Big)$