The Size of Sets

Prepared by Mark on July 23, 2023

Part 1: Set Basics

Definition 1:
A *set* is a collection of objects.
If a is an element of set S, we write $a \in S$. This is pronounced "a in S."
The position of each element in a set or the number of times it is repeated doesn't matter.
All that matters is *which* elements are in the set.

We say two sets A and B are equal if every element of A is in B, and every element of B is in A.
This is known as the *principle of extensionality.*

Problem 2:
Convince yourself that $\{a, b\} = \{b, a\} = \{a, b, a, b, b\}$.

Definition 3:
A set A is a *subset* of a set B if every element of A is in B.
For example, $\{a, b\}$ is a subset of $\{a, b, c\}$. This is written $\{a, b\} \subseteq \{a, b, c\}$.
Note that the "subset" symbol resembles the "less than or equal to" symbol.

We can also write $\{a, b\} \subset \{a, b, c\}$, which denotes a *strict subset.*
The relationship between $\subseteq$ and $\subset$ is the same as the relationship between $\leq$ and $<$.
In particular, if $A \subset B$, $A \subseteq B$ and $A \neq B$
For example, $\{a, b, c\} \subseteq \{a, b, c\}$ is true, but $\{a, b, c\} \subset \{a, b, c\}$ is false.

Definition 4:
The *empty set*, usually written $\varnothing$, is the unique set containing no elements.
By definition, the empty set is a subset of every set.
Note: The $\varnothing$ symbol is called "varnothing." If you'd like to know why, ask an instructor.

Problem 5:
Which of the following are true?

- $\{1, 3\} = \{3, 3, 1\}$
- $\{1, 2\} \subset \{2\}$
- $\{1, 2\} \subset \{1, 2\}$
- $\{1, 2\} \subseteq \{1, 2\}$
- $\{2\} \subseteq \{1, 2\}$
- $\varnothing \subseteq \{1, 2\}$

Problem 6:
Let A and B be sets. Convince yourself that $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

Hint: Whenever you start a proof, you should first look at definitions.
As stated on the previous page, $A = B$ if every element in A is in B and every element of B is in A.

As we saw before, the $\subseteq$ relation behaves a lot like the $\leq$ relation.
The statement above is very similar to the statement "$x \leq y$ and $y \geq x$ implies $x = y$".

Definition 7:
Let A be a set. The *power set* of A, written $\mathcal{P}(A)$, is the set of all subsets of A.

Problem 8:
What is the power set of $\{1, 2, 3\}$?
Hint: It has eight elements.

Problem 9:
Let A be a set with n elements.
How many elements does $\mathcal{P}(A)$ have?
Hint: Binary may help.

Problem 10:
Show that the set of all sets that do not contain themselves is not a set.

Definition 11: Set Operations
$A \cap B$ is the *intersection* of A and B.
It is the set of objects that are in both A and B.

$A \cup B$ is the *union* of A and B.
It is the set of objects that are in either A or B.

$A - B$ is the *difference* of A and B.
It is the set of objects that are in A but are not in B.

Problem 12:
What is $\{a, b, c\} \cap \{b, c, d\}$?

Problem 13:
What is $\{a, b, c\} \cup \{b, c, d\}$?

Problem 14:
What is $\{a, b, c\} - \{b, c, d\}$?

Part 2: Really Big Sets

Definition 15:
We say set is *finite* if its elements can be consecutively numbered from 1 to some maximum index n. Informally, we could say that a set is finite if it "ends."
For example, the set $\{\star, \diamond, \heartsuit\}$ is (obviously) finite. We can number its elements 1, 2, and 3.

If a set is not finite, we say it is *infinite*.

Problem 16:
Which of the following sets are finite?
- $\{\texttt{A}, \texttt{B}, ..., \texttt{Z}\}$
- {all rats in Europe}
- {all positive numbers}
- {all rational numbers}

Remark:
Note that our definition of "infinite-ness" is based on a property of the set. Saying "a set is infinite" is much like saying "a cat is black" or "a number is even". There are many different kinds of black cats, and there are many different even numbers — some large, some small.

In general, ∞ **is not a well-defined mathematical object**[1]. Infinity is not a number. There isn't a single "infinity." Infinity is the the general concept of endlessness, used in many different contexts.

[1] In most cases. There are exceptions, but you need not worry about them for now. If you're curious, you may ask an instructor to explain. There's also a chance we'll see a well-defined "infinity" in a handout later this quarter.

Part 3: Common Sets and Cartesian Products

Definition 17:
There are a few sets we use often. They have special names:
- $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of *natural numbers*.
- $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the set of *integers*.
- $\mathbb{Q}$ is the set of *rational numbers*.
- $\mathbb{R}$ is the set of *real numbers*.

Note: $\mathbb{Z}$ is called "blackboard zee" or "big zee." Naturally, $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ have similar names. This, of course, depends on context. Sometimes "zee" is all you need.

Problem 18:
Which of the following sets contain 100?
Hint: There may be more than one answer in all the problems below.

$\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$

Problem 19:
Which of the following sets contain $\frac{1}{2}$?

$\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$

Problem 20:
Which of the following sets contain π?

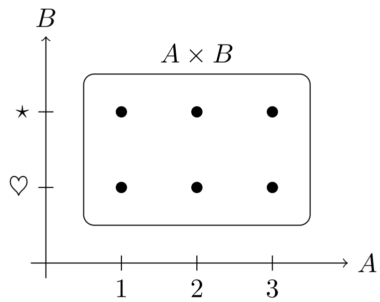
$\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$

Problem 21:
Which of the following sets contain $\sqrt{-1}$?

$\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$

Definition 22:
Consider the sets A and B. The set $A \times B$ consists of all ordered[2] pairs (a, b) where $a \in A$ and $b \in B$. This is called the *cartesian product*, and is usually pronounced "A cross B".

For example, $\{1, 2, 3\} \times \{\heartsuit, \star\} = \{(1, \heartsuit),\ (1, \star),\ (2, \heartsuit),\ (2, \star),\ (3, \heartsuit),\ (3, \star)\}$
You can think of this as placing the two sets "perpendicular" to one another:



Problem 23:
Let $A = \{0, 1\} \times \{0, 1\}$
Let $B = \{a, b\}$
What is $A \times B$?

Problem 24:
What is $\mathbb{R} \times \mathbb{R}$?
Hint: Use the "perpendicular" analogy

[2]This means that order matters. $(a, b) \neq (b, a)$.

Definition 25:
$\mathbb{R}^n$ is the set of n-tuples of real numbers.
In English, this means that an element of $\mathbb{R}^n$ is a list of n real numbers:

Elements of $\mathbb{R}^2$ look like (a, b), where $a, b \in \mathbb{R}$.
Elements of $\mathbb{R}^5$ look like $(a_1, a_2, a_3, a_4, a_5)$, where $a_n \in \mathbb{R}$.
$\mathbb{R}^1$ and $\mathbb{R}$ are identical.

Note: $\mathbb{R}^2$ is pronounced "arrgh-two."

Intuitively, $\mathbb{R}^2$ forms a two-dimensional plane, and $\mathbb{R}^3$ forms a three-dimensional space.
$\mathbb{R}^n$ is hard to visualize when $n \geq 4$, but you are welcome to try.

Problem 26:
Convince yourself that $\mathbb{R} \times \mathbb{R}$ is $\mathbb{R}^2$.
What is $\mathbb{R}^2 \times \mathbb{R}$?

Problem 27:
What is $\mathbb{N}^2$?

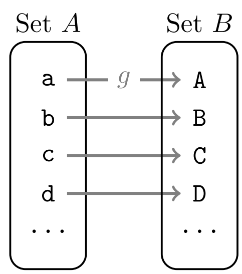
Problem 28:
What is $\mathbb{Z}^3$?

Part 4: Functions and Maps

Definition 29:
A *function* or *map* f from a set A to a set B is a rule that assigns an element of B to each element of A. We write this as $f : A \to B$.

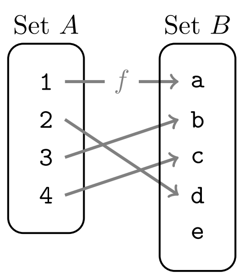
Let $L = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, ..., \mathtt{z}\}$ be the set of lowercase english letters.
Let $C = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, ..., \mathtt{Z}\}$ be the set of uppercase english letters.

Say we have a function $g : L \to C$ that capitalizes english letters.
We can think of this function as a *map* from A to B, shown below using arrows:

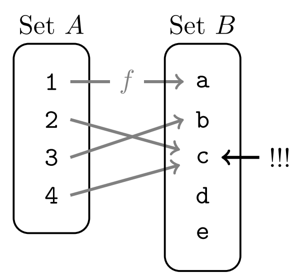


Definition 30:
We say a map f is *one-to-one* if $a = b$ implies $f(a) = f(b)$ for all $a, b \in A$.
In other words, this means that no two elements of A are mapped to the same b:

A one-to-one function:

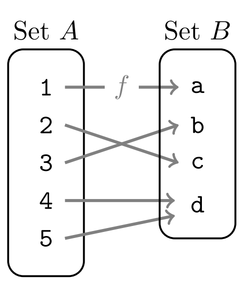


NOT a one-to-one function:

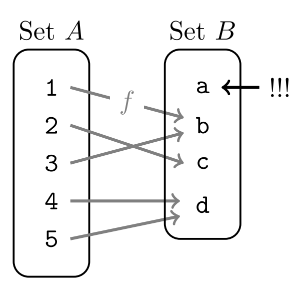


Definition 31:
We say a map f is *onto* if for every $b \in B$, there is an $a \in A$ so that $b = f(a)$.
In other words, this means that every element of B has some element of A mapped to it:

An onto function:



NOT an onto function:



Remark:
The words "function" and "map" are two views of the same mathematical object. We usually think of functions as "machines" that take an input, change it, and produce an output. We think of maps as "rules" that match each element of a set A to an element of a set B.

Again, functions and maps are *identical*. They do the same thing. The only difference between "functions" and "maps" is how we think about them.

Problem 32:
Is the "capitalize" function in Definition 29 one-to-one? Is it onto?

Problem 33:
Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$.
Is this function one-to-one? Is it onto?

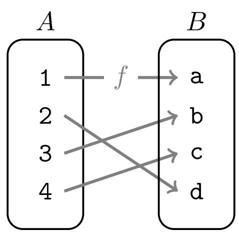
Problem 34:
Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined below.
Is this function one-to-one? Is it onto?

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x + 1 & \text{otherwise} \end{cases}$$

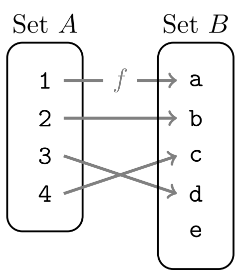
Definition 35: Invertible Functions
A function g is an *inverse* of a function f if $g(f(x)) = x$ for any x.
In other words, the function g "undoes" f. Usually, the inverse of a function f is written f^{-1}.
We say a function is *invertible* if it has an inverse.

Intuitively, we could say that the inverse of f reverses the "arrows" of f.

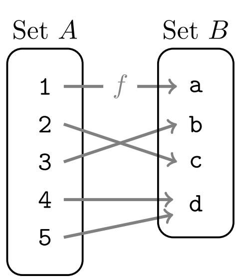
Problem 36:
Is the following function invertible?
Draw the inverse, or explain why you can't.



Problem 37:
Is the following function invertible?
Draw the inverse, or explain why you can't.



Problem 38:
Is the following function invertible?
Draw the inverse, or explain why you can't.



Definition 39: Bijections
One-to-one maps are also called *injective* maps.
Onto maps are also called *surjective* maps.

If a function is both one-to-one and onto, we say it is a *bijection*.

Theorem 40:
All bijective functions are invertible. All invertible functions are bijections.
You should review the problems on the previous page and convince yourself that this is true.

Problem 41:
We say a set S is *finite* if there exists a bijection from S to $\{1, 2, 3, ..., n\}$ for some integer n.
Convince yourself that this definition of "finite-ness" is the same as the one in Definition 15.

Problem 42:
Is there a bijection between the sets $\{1, 2, 3\}$ and $\{\texttt{A}, \texttt{B}, \texttt{C}\}$?
If a bijection exists, find one; if one doesn't, prove it.

Problem 43:
Is there a bijection between the sets $\{1, 2, 3, 4\}$ and $\{\texttt{A}, \texttt{B}, \texttt{C}\}$?
If a bijection exists, find one; if one doesn't, prove it.

Problem 44:
Let A and B be two sets of different sizes.
Show that no bijection between A and B exists.

Problem 44 reveals a very important fact: if we can find a bijection between two sets A and B, these sets must have the same number of elements. Similarly, if we know that a bijection doesn't exist, we know that A and B must have a different number of elements.

Intuitively, you can think of a bijection as a "matching" between elements of A and B. If we were to draw a bijection, we'd see an arrow connecting every element in A to every element in B. If a bijection exists, every element of A directly corresponds to an element of B, therefore A and B must have the same number of elements.

Definition 45:
We say two sets A and B are *equinumerous* if there exists a bijection $f : A \to B$.

Part 5: Enumerations

Definition 46:
Let A be a set. An *enumeration* is a bijection from A to $\{1, 2, ..., n\}$ or $\mathbb{N}$.
An enumeration assigns an element of $\mathbb{N}$ to each element of A.

Definition 47:
We say a set is *countable* if it has an enumeration.
We consider the empty set trivially countable.

Problem 48:
Find an enumeration of $\{\texttt{A}, \texttt{B}, ..., \texttt{Z}\}$.

Problem 49:
Find an enumeration of $\mathbb{N}$.

Problem 50:
Find an enumeration of the set of squares $\{1, 4, 9, 16, ...\}$.

Problem 51:
Let A and B be equinumerous sets.
Show that A is countable iff B is countable.

Problem 52:
Show that $\mathbb{Z}$ is countable.

Problem 53:
Show that $\mathbb{N}^2$ is countable.

Problem 54:
Show that $\mathbb{Q}$ is countable.

Problem 55:
Show that $\mathbb{N}^k$ is countable.

Problem 56:
Show that if A and B are countable, $A \cup B$ is also countable.
Note that this automatically solves Problem 53 and Problem 55.

Uncountable Sets

Problem 57:
Let B be the set of infinite binary strings. Show that B is not countable.
Here's how you should start:

Assume we have some enumeration $n(b)$ that assigns a natural number to every $b \in B$.
Now, arrange the elements of B in a table, in order of increasing index:

$n(b)$	digits of b
0	1010100110011110...
1	0101101011010010...
2	1101011001010101...
3	0001100101010110...
4	1101011101000110...
5	1101100010100111...
6	1011001101001010...
...	.....

First, convince yourself that if B is countable, this table will contain every element of B, then construct a new element of B that is guaranteed to *not* be in this table.
Hint: What should the first digit of this new string be? What should its second digit be? Or, even better, what *shouldn't* they be?

Problem 58:
Using Problem 57, show that $\mathcal{P}(\mathbb{N})$ is uncountable.

Problem 59:
Show that $\mathbb{R}$ is not countable.
Hint: Earlier in this handout, we defined a real number as "a decimal, finite or infinite."

Problem 60:
Find a bijection from $(0, 1)$ to $\mathbb{R}$.
Hint: $(0, 1)$ is the set of all real numbers between 0 and 1, not including either endpoint.

This problem brings us to the surprising conclusion that there are "just as many" numbers between 0 and 1 as there are in the entire real line.

Problem 61:
Find a bijection between $(0, 1)$ and $[0, 1]$.
Hint: $[0, 1]$ is the set of all real numbers between 0 and 1, including both endpoints.