Transfer Principle

Problem 1 (♦). Is there a solution for the following system of equations and inequations:

\[
\begin{align*}
x^2 + y^2 &= 0 \\
x + y &\neq 0
\end{align*}
\]

In the previous handout, we considered a nonarchimedean extension $F$ of field $\mathbb{R}$. It was enough to require that $F$ has four arithmetic operations and infinitesimals to use it to take derivatives. But even if we try to differentiate $f(x) = \sqrt{x}$, we need to make sure that in $F$ we can take square roots. If we want to take derivatives of functions like $\sin(x)$ we require that in $F$ we can take sines as well. It is not true for any extension $F$. So here we will use a very specific extension $\ast \mathbb{R}$. We call elements of $\ast \mathbb{R}$ hyperreals.

Definition 1. Any function $f(x_1, \ldots, x_n)$ of $n$ real variables has a nonstandard extension $\ast f(x_1, \ldots, x_n)$, which takes hyperreal inputs $x_1, \ldots, x_n$ and outputs a hyperreal, such that it coincides with $f$ on all real inputs.

Definition 2 (Transfer Principle). Any system of equations and inequations using functions has a solution in $\mathbb{R}$ if and only if the $\ast$-analog of this system has a solution in $\ast \mathbb{R}$.

Let us use the transfer principle to get some consequences:

Example 1. If $f : \mathbb{R} \to \mathbb{R}$ only takes values 0 and 1, then $\ast f$ also only takes values 0 and 1.

Proof. Consider the system

\[
\begin{align*}
f(x) &\neq 0, \\
f(x) &\neq 1.
\end{align*}
\]

Since it has no solutions in reals, its $\ast$-analog

\[
\begin{align*}
\ast f(x) &\neq 0, \\
\ast f(x) &\neq 1.
\end{align*}
\]

has no solutions in hyperreals. So $\ast f$ also only takes values 0 and 1. □
Remark 1. We remind you that the possibility of the existence of such a nice field $^\ast\mathbb{R}$ is a hypothesis for now. We will work with the consequences of its existence for the whole course and will prove it if time permits.

**Theorem 1** (Abraham Robinson). Such field $^\ast\mathbb{R}$ exists.

**Example 2.** If $f$ and $g$ are functions such that sets of their zeroes coincide, then for $^\ast f$ and $^\ast g$ their sets of zeroes coincide.

**Proof.** Indeed, systems

\begin{align*}
  f(x) = 0 \text{ and } g(x) \neq 0 \\
  f(x) \neq 0 \text{ and } g(x) = 0
\end{align*}

have no solutions, so the systems

\begin{align*}
  ^\ast f(x) = 0 \text{ and } ^\ast g(x) \neq 0 \\
  ^\ast f(x) \neq 0 \text{ and } ^\ast g(x) = 0
\end{align*}

also have no solutions. □

**Problem 2.** Prove that $^\ast \sin(x)$ is never equal to 42.

**Definition 3.** Examples 1 and 2 allow us to define an enlargement $^\ast A$ for any set $A \subseteq \mathbb{R}$. Indeed, consider any function $f$ which has $A$ as its set of zeroes (such a function exists, e.g.

\[ f(x) = \begin{cases} 
  0, & \text{if } x \in A \\
  1, & \text{otherwise.}
\end{cases} \]

suits). Then $^\ast A$ is a set of zeroes of $^\ast f$. Note that by Example 2 this definition is independent of $f$.

**Problem 3.** Show that $^\ast \emptyset = \emptyset$.

**Problem 4** (♀). Prove that if $A = B \cap C$, then $^\ast A = ^\ast B \cap ^\ast C$.

**Problem 5** (♀). Prove that any number $n \in ^\ast \mathbb{N}$ also belongs to $^\ast \mathbb{Q}$.

**Example 3.** Any set $A$ is a subset of $^\ast A$. So the term “enlargement” is justified.

**Proof.** Let $a$ be an element of $A$ and $f$ be some function with $A$ as a set of zeroes. Then $^\ast f(a) = 0$ since $^\ast f$ coincides with $f$ on real inputs. So $a$ belongs to $^\ast \mathbb{R}$. □

**Problem 6** (♀). a) Show that if $A = \{0, 1\}$, then $^\ast A = A$.

b) Show that if $A$ is finite, then $^\ast A = A$.

The transfer principle also works with functions of $2$ and more arguments. Note that the transfer principle is enough to see that $^\ast \mathbb{R}$ is an ordered field. For example, let’s check the fifth axiom of an ordered field.

**Example 4.** For any $x, y, z \in ^\ast \mathbb{R}$ one has $(x + y)z = xz + yz$. 

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Proof. Indeed, addition and multiplication are just functions taking two arguments. We can write them as \(a(x, y) = x + y\) and \(m(x, y) = xy\) to remember it. Then the inequality
\[
m(a(x, y), z) \neq a(m(x, z), m(x, z))
\]
has no solutions in reals, so hyperaddition \(^a\) and hypermultiplication \(^m\) also never fail the fifth axiom.

Problem 7. a) Show that (hyper)products of hyperrational numbers are also hyperrational.

b) (*) Can a sum of two hyperirrational numbers be hyperrational?

Problem 8. (It will be used in problem \[12\]) Show that
\[
^*\sqrt(x) + ^*\sqrt(y) = (x \cdot y) ^*/ (^*\sqrt(x) + ^*\sqrt(y)),
\]
where \(\sqrt(x) = \sqrt{x}\).

Remark 2. In the sequel, we will omit stars while using functions, so we just write \(f\) instead of \(^*f\) and \(+, -, \sqrt, \sin, \ln, |x|\) instead of \(^*+, ^*-, ^*\sqrt, ^*\sin, ^*\ln, ^*\abs(x)\) and so on.

Problem 9 (♀).
\[
\text{less}(x, y) = \begin{cases} 
1, & \text{if } x < y, \\
0, & \text{otherwise}.
\end{cases}
\]
Check that the axiom 11 holds for \(^*\mathbb{R}\): for any \(x, y, z \in ^*\mathbb{R}\) such that \(\text{less}(x, y) = 1\) and \(\text{less}(y, z) = 1\), we also have \(\text{less}(x, z) = 1\).

Definition 4. A binary relation on set \(X\) is a function from \(X \times X\) to \{True, False\}. Examples of relations on \(\mathbb{R}\) are =, \(\neq\) and <. On \(\mathbb{N}\) there is a relation “divides”, which is usually denoted by \(\mid\) (so \(7\mid98\)).

For any binary relation \(R(x, y)\) on \(^*\mathbb{R}\), one may define a binary relation \(^R\) on \(^*\mathbb{R}\) using the same trick as the previous problem does with <.

In the sequel, we will write \(R\) instead of \(^R\), as we strictly speaking should.

Problem 10 (♀). a) Show that if \(y > x + 1\) for \(x, y \in ^*\mathbb{R}\), then there exists an integer \(n\) s.t. \(x < n < y\).

b) Show that between any two different hyperreals there is a hyperrational.

c) Show that any hyperreal is infinitely close to some hyperrational.

Remark 3. In particular, since \(^*\mathbb{R}\) is nonarchimedean, there exists some positive unlimited hyperreal, so there is an unlimited hypernatural.

Recall that \(x \in ^*\mathbb{R}\) is called unlimited if \(|x|\) is greater than any standard number and infinitesimal if \(|x|\) is smaller than every standard positive real.

Problem 11. Let \(\abs(x) = |x|\) be the absolute value function. Prove that \(^*\abs(x)\) coincides with \(|x|\) as defined for any nonarchimedean extension in the previous worksheet.
**Problem 12.**

a) Show that for any positive unlimited \( H \) the square root of \( H \) is also unlimited.

b) Show that for any positive unlimited \( H \) the number \( \sqrt{H + 1} - \sqrt{H} \) is infinitesimal.

**Problem 13 (gradable).**

a) Find a function \( f \), which is not identically zero, such that \( f(x) = 0 \) for all unlimited \( x \);

b) (*) Find an increasing function \( f \), which is not a constant, such that \( f(2H) - f(H) \) is infinitesimal for all positive unlimited \( H \);

**Problem 14 (gradable).**

Develop a theory of prime factors in \( {}^*\mathbb{N} \): if \( P \) is the set of standard prime numbers, with enlargement \( {}^*P \subset {}^*\mathbb{N} \), prove the following:

a) Show that for any \( M \in {}^*\mathbb{N} \) there is an \( N \in {}^*\mathbb{N} \) that is divisible in \( {}^*\mathbb{N} \) by all members of \( \{1, 2, \ldots, M\} \). Hence show that there exists a hypernatural number \( N \) that is divisible by every standard positive integer.

b) \( {}^*P \) consists precisely of those hypernaturals \( > 1 \) that have no nontrivial factors in \( {}^*\mathbb{N} \).

c) Every hypernatural number \( > 1 \) has a "hyperprime" factor, i.e., is divisible by some member of \( {}^*P \).

d) Show that there exists an unlimited hyperprime (Hint: see Remark 3)

Recall that a real number \( x_0 \) is said to be the standard part of \( x \), denoted \( st(x) \), if it is infinitely close to \( x \). A hyperreal is called standard, if it is a real, and nonstandard, if it is not.

**Problem 15 (gradable).**

a) Show that for any infinite set \( A \) there is a nonstandard element of \( {}^*A \);

b) Show that for any unbounded set \( A \) there is an unlimited element of \( {}^*A \).

**Problem 16.**

a) Show that for any bounded set \( A \) there is an element \( x \in {}^*A \), which is greater or equal than any standard \( y \in A \).

b) Show that for any such \( x \) one has \( st(x) = sup A \).

**Problem 17 (gradable).** Let \( A \) and \( B \) be subsets of \( \mathbb{R} \), and let \( sup A \) and \( sup B \) be known.

1. Find \( sup(A \cup B) \).

2. Find \( sup(A + B) \), where \( A + B = \{a + b \mid a \in A, b \in B\} \).

3. Find \( inf(A \cdot B) \), where \( A \cdot B = \{a \cdot b \mid a \in A, b \in B\} \), if \( A \) and \( B \) consist of negative numbers.

**Remark 4.** Note that the proof involves no \( \varepsilon \)-guessing, an annoying technique prevalent in standard approach to analysis, which we discussed on the second meeting.