Nonstandard analysis

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The conduit is available at `https://tinyurl.com/ORMCconduit`.

Transfer Principle

Problem 1 (✎). Is there a solution for the following system of equations and inequations:

$$\begin{cases} x^2 + y^2 = 0 \\ x + y \neq 0 \end{cases} \quad ?$$

In the previous handout, we considered a nonarchimedean extension F of field $\mathbb{R}$. It was enough to require that F has four arithmetic operations and infinitesimals to use it to take derivatives. But even if we try to differentiate $f(x) = \sqrt{x}$, we need to make sure that in F we can take square roots. If we want to take derivatives of functions like $\sin(x)$, we require that in F we can take sines as well. It is not true for *any* extension F. So here we will use a very specific extension ${}^*\mathbb{R}$. We call elements of ${}^*\mathbb{R}$ the *hyperreals*.

Definition 1. Any function $f(x_1, \ldots, x_n)$ of n real variables has a *nonstandard extension* ${}^*f(x_1, \ldots, x_n)$, which takes hyperreal inputs $x_1, \ldots, x_n$ and outputs a hyperreal, such that it coincides with f on all real inputs.

Definition 2 (Transfer Principle)**.** Any system of equations and inequations using functions has a solution in $\mathbb{R}$ if and only if the *-analog of this system has a solution in ${}^*\mathbb{R}$.

Let us use the transfer principle to get some consequences:

Example 1. If $f : \mathbb{R} \to \mathbb{R}$ only takes values 0 and 1, then ${}^*f$ also only takes values 0 and 1.

Proof. Consider the system

$$\begin{cases} f(x) \neq 0, \\ f(x) \neq 1. \end{cases}$$

Since it has no solutions in reals, its *-analog

$$\begin{cases} {}^*f(x) \neq 0, \\ {}^*f(x) \neq 1. \end{cases}$$

has no solutions in hyperreals. So ${}^*f$ also only takes values 0 and 1. □

Remark 1. We remind you that the possibility of the existence of such a nice field ${}^*\mathbb{R}$ is a hypothesis for now. We will work with the consequences of its existence for the whole course and will prove it if time permits.

Theorem 1 (Abraham Robinson)**.** *Such field ${}^*\mathbb{R}$ exists.*

Example 2. If f and g are functions such that sets of their zeroes coincide, then for ${}^*f$ and ${}^*g$ their sets of zeroes coincide.

Proof. Indeed, the systems

$$f(x) = 0 \text{ and } g(x) \neq 0 \tag{1}$$

$$f(x) \neq 0 \text{ and } g(x) = 0 \tag{2}$$

have no solutions, so the systems

$$^*f(x) = 0 \text{ and } {}^*g(x) \neq 0 \tag{3}$$

$$^*f(x) \neq 0 \text{ and } {}^*g(x) = 0 \tag{4}$$

also have no solutions. □

Problem 2. Prove that ${}^*\sin(x)$ is never equal to 42.

Definition 3. Examples 1 and 2 allow us to define an enlargement ${}^*A$ for any set $A \subseteq \mathbb{R}$. Indeed, consider any function f which has A as the set of its zeroes (such a function exists, e.g.

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{otherwise.} \end{cases}$$

suits). Then ${}^*A$ is a set of zeroes of ${}^*f$. Note that by Example 2 this definition is independent of f.

Problem 3. Show that ${}^*\varnothing = \varnothing$.

Problem 4 (✎)**.** Prove that if $A = B \cap C$, then ${}^*A = {}^*B \cap {}^*C$.

Problem 5 (✎)**.** Prove that any number $n \in {}^*\mathbb{N}$ also belongs to ${}^*\mathbb{Q}$.

Example 3. Any set A is a subset of ${}^*A$. So the term "enlargement" is justified.

Proof. Let a be an element of A and f be some function with A as a set of zeros. Then ${}^*f(a) = 0$ since ${}^*f$ coincides with f on real inputs. So a belongs to ${}^*A$. □

Problem 6 (✎)**.** a) Show that if $A = \{0, 1\}$, then ${}^*A = A$.

b) Show that if A is finite, then ${}^*A = A$.

The transfer principle also works with functions of 2 and more arguments. Note that the transfer principle is enough to see that ${}^*\mathbb{R}$ is an ordered field. For example, let's check the fifth axiom of an ordered field.

Example 4. For any $x, y, z \in {}^*\mathbb{R}$ one has $(x + y)z = xz + yz$.

Proof. Indeed, addition and multiplication are just functions taking two arguments. We can write them as $a(x, y) = x + y$ and $m(x, y) = xy$ to remember it. Then the inequality

$$m(a(x, y), z) \neq a(m(x, z), m(x, z))$$

has no solutions in reals, so hyperaddition ${}^*a$ and hypermultiplication ${}^*m$ also never fail the fifth axiom. □

Problem 7. a) Show that (hyper)products of hyperrational numbers are also hyperrational.

b) (*) Can a sum of two hyperirrational numbers be hyperrational?

Problem 8. (It will be used in problem 12) Show that

$${}^*\mathrm{sqrt}(x) \;{}^*\!- {}^*\mathrm{sqrt}(y) = (x \;{}^*\!- y) \;{}^*\!/ \; ({}^*\mathrm{sqrt}(x) \;{}^*\!+ {}^*\mathrm{sqrt}(y)),$$

where $\mathrm{sqrt}(x) = \sqrt{x}$.

> **Remark 2.** In the sequel, we will omit stars while using functions, so we just write f instead of ${}^*f$ and $+, -, \sqrt{}, \sin, \ln, |x|$ instead of ${}^*+, {}^*-, {}^*\mathrm{sqrt}, {}^*\sin, {}^*\ln, {}^*\mathrm{abs}(x)$ and so on.

Problem 9 (✎).

$$\mathrm{less}(x, y) = \begin{cases} 1, & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Check that the axiom 11 holds for ${}^*\mathbb{R}$: for any $x, y, z \in {}^*\mathbb{R}$ such that $\mathrm{less}(x, y) = 1$ and $\mathrm{less}(y, z) = 1$, we also have $\mathrm{less}(x, z) = 1$.

> **Definition 4.** A *binary relation* on set X is a function from $X \times X$ to $\{\mathrm{True}, \mathrm{False}\}$. Examples of relations on $\mathbb{R}$ are $=$, $\neq$ and $<$. On $\mathbb{N}$ there is a relation "divides", which is usually denoted by $|$ (so $7 \,|\, 98$).
>
> For any binary relation $R(x, y)$ on ${}^*\mathbb{R}$, one may define a binary relation ${}^*R$ on ${}^*\mathbb{R}$ using the same trick as the previous problem does with $<$.
>
> In the sequel, we will write R instead of ${}^*R$, as we strictly speaking should.

Problem 10 (✎). a) Show that if $y > x + 1$ for $x, y \in {}^*\mathbb{R}$, then there exists an integer n s.t. $x < n < y$.

b) Show that between any two different hyperreals there is a hyperrational.

c) Show that any hyperreal is infinitely close to some hyperrational.

> **Remark 3.** In particular, since ${}^*\mathbb{R}$ is nonarchimedean, there exists some positive unlimited hyperreal, so there is an unlimited hypernatural.

Recall that $x \in {}^*\mathbb{R}$ is called *unlimited* if $|x|$ is greater than any standard number and *infinitesimal.* if $|x|$ is smaller than every standard positive real.

Problem 11. Let $\mathrm{abs}(x) = |x|$ be the absolute value function. Prove that ${}^*\mathrm{abs}(x)$ coincides with $|x|$ as defined for any nonarchimedean extension in the previous worksheet.

Problem 12. a) Show that for any positive unlimited H the square root of H is also unlimited.

b) Show that for any positive unlimited H the number $\sqrt{H+1} - \sqrt{H}$ is infinitesimal.

Problem 13 (✐). a) Find a function f, which is not identically zero, such that $f(x) = 0$ for all unlimited x;

b) (*) Find an increasing function f, which is not a constant, such that $f(2H) - f(H)$ is infinitesimal for all positive unlimited H;

Problem 14 (*). Develop a theory of prime factors in ${}^*\mathbb{N}$: if $\mathbb{P}$ is the set of standard prime numbers, with enlargement ${}^*\mathbb{P} \subset^* \mathbb{N}$, prove the following:

a) Show that for any $M \in {}^*\mathbb{N}$ there is an $N \in {}^*\mathbb{N}$ that is divisible in ${}^*\mathbb{N}$ by all members of $\{1, 2, \dots, M\}$. Hence show that there exists a hypernatural number N that is divisible by every standard positive integer.

b) ${}^*\mathbb{P}$ consists precisely of those hypernaturals > 1 that have no nontrivial factors in ${}^*\mathbb{N}$.

c) Every hypernatural number > 1 has a "hyperprime" factor, i.e., is divisible by some member of ${}^*\mathbb{P}$.

d) Show that there exists an unlimited hyperprime *(Hint: see Remark 3)*

Recall that a *real* number x_0 is said to be the *standard part* of x, denoted $\mathrm{st}(x)$, if it is infinitely close to x. A hyperreal is called *standard*, if it is a real, and *nonstandard*, if it is not.

Problem 15 (✐). a) Show that for any infinite set A there is a nonstandard element of ${}^*A$;

b) Show that for any unbounded set A there is an unlimited element of ${}^*A$.

Problem 16. a) Show that for any bounded set A there is an element $x \in {}^*A$, which is greater or equal than any standard $y \in A$.

b) Show that for any such x one has $\mathrm{st}(x) = \sup A$.

Problem 17 (✐). Let A and B be subsets of $\mathbb{R}$, and let $\sup A$ and $\sup B$ be known.

1. Find $\sup(A \cup B)$.
2. Find $\sup(A + B)$, where $A + B = \{a + b \mid a \in A, b \in B\}$.
3. Find $\inf(A \cdot B)$, where $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$, if A and B consist of negative numbers.

Remark 4. Note that the proof involves no ε-guessing, an annoying technique prevalent in the standard approach to analysis, which we discussed in the second meeting.