Nonstandard analysis

Nikita and Stepan

Summer 2023

The conduit is available at https://tinyurl.com/ORMCconduit

Transfer Principle

Problem 1 (★). Is there a solution for the following system of equations and inequations:

\[
\begin{align*}
x^2 + y^2 &= 0 \\
x + y &\neq 0
\end{align*}
\]

In the previous handout, we considered a nonarchimedean extension \( F \) of field \( \mathbb{R} \). It was enough to require that \( F \) has four arithmetic operations and infinitesimals to use it to take derivatives. But even if we try to differentiate \( f(x) = \sqrt{x} \), we need to make sure that in \( F \) we can take square roots. If we want to take derivatives of functions like \( \sin(x) \), we require that in \( F \) we can take sines as well. It is not true for any extension \( F \). So here we will use a very specific extension \( {}^*\mathbb{R} \). We call elements of \( {}^*\mathbb{R} \) the hyperreals.

Definition 1. Any function \( f(x_1, \ldots, x_n) \) of \( n \) real variables has a nonstandard extension \( {}^*f(x_1, \ldots, x_n) \), which takes hyperreal inputs \( x_1, \ldots, x_n \) and outputs a hyperreal, such that it coincides with \( f \) on all real inputs.

Definition 2 (Transfer Principle). Any system of equations and inequations using functions has a solution in \( \mathbb{R} \) if and only if the \( {}^*\)-analog of this system has a solution in \( {}^*\mathbb{R} \).

Let us use the transfer principle to get some consequences:

Example 1. If \( f : \mathbb{R} \rightarrow \mathbb{R} \) only takes values 0 and 1, then \( {}^*f \) also only takes values 0 and 1.

Proof. Consider the system

\[
\begin{align*}
f(x) &\neq 0, \\
f(x) &\neq 1.
\end{align*}
\]

Since it has no solutions in reals, its \( {}^*\)-analog

\[
\begin{align*}
{}^*f(x) &\neq 0, \\
{}^*f(x) &\neq 1.
\end{align*}
\]

has no solutions in hyperreals. So \( {}^*f \) also only takes values 0 and 1.
**Remark 1.** We remind you that the possibility of the existence of such a nice field \(^*\mathbb{R}\) is a hypothesis for now. We will work with the consequences of its existence for the whole course and will prove it if time permits.

**Theorem 1** (Abraham Robinson). Such field \(^*\mathbb{R}\) exists.

**Example 2.** If \(f\) and \(g\) are functions such that sets of their zeroes coincide, then for \(^*f\) and \(^*g\) their sets of zeroes coincide.

**Proof.** Indeed, the systems

\[
\begin{align*}
  f(x) &= 0 \text{ and } g(x) \neq 0 \\
  f(x) &\neq 0 \text{ and } g(x) = 0
\end{align*}
\]

have no solutions, so the systems

\[
\begin{align*}
  ^*f(x) &= 0 \text{ and } ^*g(x) \neq 0 \\
  ^*f(x) &\neq 0 \text{ and } ^*g(x) = 0
\end{align*}
\]

also have no solutions.

**Problem 2.** Prove that \(^*\sin(x)\) is never equal to 42.

**Definition 3.** Examples 1 and 2 allow us to define an enlargement \(^*A\) for any set \(A \subseteq \mathbb{R}\). Indeed, consider any function \(f\) which has \(A\) as the set of its zeroes (such a function exists, e.g.

\[
  f(x) = \begin{cases} 
    0, & \text{if } x \in A \\
    1, & \text{otherwise.}
  \end{cases}
\]

suits). Then \(^*A\) is a set of zeroes of \(^*f\). Note that by Example 2 this definition is independent of \(f\).

**Problem 3.** Show that \(^*\emptyset = \emptyset\).

**Problem 4** (♀). Prove that if \(A = B \cap C\), then \(^*A = ^*B \cap ^*C\).

**Problem 5** (♂). Prove that any number \(n \in ^*\mathbb{N}\) also belongs to \(^*\mathbb{Q}\).

**Example 3.** Any set \(A\) is a subset of \(^*A\). So the term “enlargement” is justified.

**Proof.** Let \(a\) be an element of \(A\) and \(f\) be some function with \(A\) as a set of zeroes. Then \(^*f(a) = 0\) since \(^*f\) coincides with \(f\) on real inputs. So \(a\) belongs to \(^*A\). 

**Problem 6** (♀).  
  a) Show that if \(A = \{0, 1\}\), then \(^*A = A\).

  b) Show that if \(A\) is finite, then \(^*A = A\).

The transfer principle also works with functions of 2 and more arguments. Note that the transfer principle is enough to see that \(^*\mathbb{R}\) is an ordered field. For example, let’s check the fifth axiom of an ordered field.

**Example 4.** For any \(x, y, z \in ^*\mathbb{R}\) one has \((x + y)z = xz + yz\).
Indeed, addition and multiplication are just functions taking two arguments. We can write them as \( a(x, y) = x + y \) and \( m(x, y) = xy \) to remember it. Then the inequality
\[
m(a(x, y), z) \neq a(m(x, z), m(x, z))
\]
has no solutions in reals, so hyperaddition \( \ast a \) and hypermultiplication \( \ast m \) also never fail the fifth axiom.

**Problem 7.**

a) Show that (hyper)products of hyperrational numbers are also hyperrational.

b) (*) Can a sum of two hyperirrational numbers be hyperrational?

**Problem 8.** (It will be used in problem 12) Show that
\[
\ast \sqrt(x) \ast - \ast \sqrt(y) = (x \ast - y) \ast (\ast \sqrt(x) \ast + \ast \sqrt(y)),
\]
where \( \sqrt(x) = \sqrt{x} \).

**Remark 2.** In the sequel, we will omit stars while using functions, so we just write \( f \) instead of \( \ast f \) and \( +, -, \sqrt, \sin, \ln, |x| \) instead of \( \ast +, \ast -, \ast \sqrt, \ast \sin, \ast \ln, \ast |x| \) and so on.

**Problem 9 (♀).**

\[
\text{less}(x, y) = \begin{cases} 
1, & \text{if } x < y, \\
0, & \text{otherwise}.
\end{cases}
\]

Check that the axiom 11 holds for \( \ast R \): for any \( x, y, z \in \ast R \) such that \( \text{less}(x, y) = 1 \) and \( \text{less}(y, z) = 1 \), we also have \( \text{less}(x, z) = 1 \).

**Definition 4.** A binary relation on set \( X \) is a function from \( X \times X \) to \{True, False\}. Examples of relations on \( \mathbb{R} \) are \( =, \neq \) and \( < \). On \( \mathbb{N} \) there is a relation “divides”, which is usually denoted by \( | \) (so \( 7 | 98 \)).

For any binary relation \( R(x, y) \) on \( \ast \mathbb{R} \), one may define a binary relation \( \ast R \) on \( \ast \mathbb{R} \) using the same trick as the previous problem does with \( < \).

In the sequel, we will write \( R \) instead of \( \ast R \), as we strictly speaking should.

**Problem 10 (♀).**

a) Show that if \( y > x + 1 \) for \( x, y \in \ast \mathbb{R} \), then there exists an integer \( n \) s.t. \( x < n < y \).

b) Show that between any two different hyperreals there is a hyperrational.

c) Show that any hyperreal is infinitely close to some hyperrational.

**Remark 3.** In particular, since \( \ast \mathbb{R} \) is nonarchimedean, there exists some positive unlimited hyperreal, so there is an unlimited hypernatural.

Recall that \( x \in \ast \mathbb{R} \) is called unlimited if \( |x| \) is greater than any standard number and infinitesimal if \( |x| \) is smaller than every standard positive real.

**Problem 11.** Let \( \text{abs}(x) = |x| \) be the absolute value function. Prove that \( \ast \text{abs}(x) \) coincides with \( |x| \) as defined for any nonarchimedean extension in the previous worksheet.
Problem 12. a) Show that for any positive unlimited $H$ the square root of $H$ is also unlimited.

b) Show that for any positive unlimited $H$ the number $\sqrt{H} + 1 - \sqrt{H}$ is infinitesimal.

Problem 13 (🔗). a) Find a function $f$, which is not identically zero, such that $f(x) = 0$ for all unlimited $x$;

b) (*) Find an increasing function $f$, which is not a constant, such that $f(2H) - f(H)$ is infinitesimal for all positive unlimited $H$;

Problem 14 (*). Develop a theory of prime factors in $^*\mathbb{N}$ : if $P$ is the set of standard prime numbers, with enlargement $^*P \subset ^*\mathbb{N}$, prove the following:

a) Show that for any $M \in ^*\mathbb{N}$ there is an $N \in ^*\mathbb{N}$ that is divisible in $^*\mathbb{N}$ by all members of $\{1, 2, \ldots, M\}$. Hence show that there exists a hypernatural number $N$ that is divisible by every standard positive integer.

b) $^*P$ consists precisely of those hypernaturals $> 1$ that have no nontrivial factors in $^*\mathbb{N}$.

c) Every hypernatural number $> 1$ has a "hyperprime" factor, i.e., is divisible by some member of $^*P$.

d) Show that there exists an unlimited hyperprime (Hint: see Remark 3)

Recall that a real number $x_0$ is said to be the standard part of $x$, denoted $st(x)$, if it is infinitely close to $x$. A hyperreal is called standard, if it is a real, and nonstandard, if it is not.

Problem 15 (🔗). a) Show that for any infinite set $A$ there is a nonstandard element of $^*A$;

b) Show that for any unbounded set $A$ there is an unlimited element of $^*A$.

Problem 16. a) Show that for any bounded set $A$ there is an element $x \in ^*A$, which is greater or equal than any standard $y \in A$.

b) Show that for any such $x$ one has $st(x) = sup A$.

Problem 17 (🔗). Let $A$ and $B$ be subsets of $\mathbb{R}$, and let $sup A$ and $sup B$ be known.

1. Find $sup(A \cup B)$.

2. Find $sup(A + B)$, where $A + B = \{a + b \mid a \in A, b \in B\}$.

3. Find $inf(A \cdot B)$, where $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$, if $A$ and $B$ consist of negative numbers.

Remark 4. Note that the proof involves no $\varepsilon$-guessing, an annoying technique prevalent in the standard approach to analysis, which we discussed in the second meeting.