ORMC Intermediate I **July 16, 2023**

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The following is the list of the important Boolean algebra properties we have learned.

Addition is commutative.

$$A + B = B + A \tag{1}$$

Multiplication is commutative.

$$A \times B = B \times A \tag{2}$$

Addition is associative.

$$(A + B) + C = A + (B + C) \tag{3}$$

Multiplication is associative.

$$(A \times B) \times C = A \times (B \times C) \tag{4}$$

Multiplication is distributive with respect to addition.

$$A \times (B + C) = (A \times B) + (A \times C) \tag{5}$$

If the above formulas look familiar to nearly every middle school student, then most of the below are Boolean algebra specific.

Addition is distributive with respect to multiplication.

$$A + (B \times C) = (A + B) \times (A + C) \tag{6}$$

Identity Law

$$A + 0 = A \quad \text{and} \quad A + 1 = 1 \tag{7}$$

$$A \times 0 = 0 \quad \text{and} \quad A \times 1 = A \tag{8}$$

Idempotent Law

$$\underbrace{A + A + \ldots + A}_{\text{n times}} = A \tag{9}$$

$$\underbrace{A \times A \times \ldots \times A}_{\text{n times}} = A \tag{10}$$

Involution Law

$$\neg\neg A = A \tag{11}$$

The principle of the excluded middle a.k.a. the law of the excluded third: either A or $\neg A$ must be true.

$$A + \neg A = 1 \tag{12}$$

A and $\neg A$ cannot be true simultaneously.

$$A \times \neg A = 0 \tag{13}$$

Boolean algebra addition and multiplication are dual with respect to negation. The following two formulas showing it are known as De Morgan's laws.

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \tag{14}$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \tag{15}$$

The following two identities are known as the laws of *absorption*.

$$A + AB = A \tag{16}$$

$$A \times (A + B) = A \tag{17}$$

Recall that to simplify a Boolean algebra expression means to find an equivalent expression that contains no negations of composite statements and uses as few variables as possible.

Problem 1 *Simplify the following expressions.*

- $\neg(AB + C\neg D) + \neg B + \neg C =$

- $A + A(\neg(B\neg C + \neg BD)) =$

- $X + XY + XYZ + X\neg YZ + XY\neg Z =$

Boolean algebra logic problems

To solve the following logic problems, we can use Boolean algebra. To do so, give a name (assign a variable) to every simple statement in a problem. Then translate all the available information into some Boolean algebra formulas using the operations $+$, $\times$, and $\neg$. If possible, try to boil everything down to one formula. Simplify the formula and see if the simplified version makes the solution obvious.

Problem 36 from the previous handout *The year is 3014. Four kids got to the final tour of GMC8 (Galactic Math Olympiad for 8th graders): Nathan, Michelle, Laura, and Reinhardt. Some knowledgeable ORMC fans discussed their chances to win. One student thought that Nathan would take first place and Michelle would take second. Another student thought that Laura would take second place while Reinhardt would end up the last of the four. The third student thought that Nathan would be second and Reinhardt third. When the results of the competition came out, it turned out that each of the ORMC students had made only one of their two predictions correctly. Find the places Nathan, Michelle, Laura, and Reinhardt got at GMC8-3014.*

Solution The following are the simple statements made above.

- N_1 = *Nathan takes the first place.*
- M_2 = *Michelle takes the second place.*
- L_2 = *Laura takes the second place.*
- R_4 = *Reinhardt takes the fourth place.*
- N_2 = *Nathan takes the second place.*
- R_3 = *Reinhardt takes the third place.*

Let us use the simple statements above to translate the story into the Boolean algebra language. The first fan made a composite statement N_1M_2 that turned out to be false.

$$N_1M_2 = 0$$

The fact that a half of the guess is true means that either $N_1\neg M_2 = 1$ and $\neg N_1M_2 = 0$ or that $N_1\neg M_2 = 0$ and $\neg N_1M_2 = 1$. This can be expressed by means of a single formula.

$$N_1\neg M_2 + \neg N_1M_2 = 1 \tag{18}$$

A similar translation of the other two fans' predictions into the Boolean algebra language gives us the following.

$$L_2\neg R_4 + \neg L_2R_4 = 1 \tag{19}$$

$$N_2\neg R_3 + \neg N_2R_3 = 1 \tag{20}$$

Multiplying 18, 19, and 20 brings together all the information we have about the competition.

$$(N_1\neg M_2+\neg N_1 M_2)(L_2\neg R_4+\neg L_2 R_4)(N_2\neg R_3+\neg N_2 R_3) = 1 \quad (21)$$

Let us first find the product of the second and third factors.

$$(L_2\neg R_4 + \neg L_2 R_4)(N_2\neg R_3 + \neg N_2 R_3) = 1$$

Opening parentheses gives the following.

$$L_2\neg R_4 N_2\neg R_3+L_2\neg R_4\neg N_2 R_3+\neg L_2 R_4 N_2\neg R_3+\neg L_2 R_4\neg N_2 R_3 = 1$$

Since Laura and Nathan cannot take the second place simultaneously, $L_2\neg R_4 N_2\neg R_3 = 0$. Since Reinhardt cannot take the third and fourth place at the same time, $\neg L_2 R_4\neg N_2 R_3 = 0$. The above sum shortens to just two terms.

$$L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3 = 1$$

This way, 21 boils down to the following.

$$(N_1\neg M_2 + \neg N_1 M_2)(L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3) = 1$$

$$N_1\neg M_2 L_2\neg R_4\neg N_2 R_3+N_1\neg M_2\neg L_2 R_4 N_2\neg R_3+\neg N_1 M_2 L_2\neg R_4\neg N_2 R_3+ \neg N_1 M_2\neg L_2 R_4 N_2\neg R_3 = 1$$

Since $N_1 N_2 = 0$, the second term is equal to zero. Since $M_2 L_2 = 0$, the third term is equal to zero as well. Since $M_2 N_2 = 0$, the same is true for the last term. We end up with the equation

$$N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 = 1$$

that tells us the results of the competition. Nathan takes the first place, Laura the second, Reinhardt the third. Therefore, Michelle takes the fourth place. There are no contradictions: Michelle is not second, Reinhardt is not fourth, and Nathan is not second. We have solved the problem!

The following is problem 37 from the previous handout. It is not hard to solve without using the algebra of logic. However, we will use the problem to test the power of Boolean algebra.

Problem 2 *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many possible schedules are there?*

Problems 3-5 are based on the following setup: Once upon a time in a land far, far away, there lived a king who invented the following way of punishing criminals. Convicted lawbreakers were given a choice between two doors. Behind each door, there could be either a hungry tiger or a treasure of gold, but not nothing or both. The king would also post some warnings on the doors and then let the criminals choose.

Problem 3 *The king took the prisoner to the doors. There was a sign on each door. The first read, "There is gold in this room and there is a tiger in the other." The sign on the second door read, "There is gold in one of these rooms and in one of these rooms there is a tiger." "Are the signs true?" asked the prisoner. "One of them is," replied the king, "but the other is not. Now, make your choice, buddy!" Which door should the prisoner open? Why?*

Problem 4 *For the second prisoner, the following signs were put on the doors. Door 1: At least one of these rooms contains gold. Door 2: A tiger is in the other room. "Are the signs true?" asked the prisoner. "They are either both true or both false," replied the king. Which door should the prisoner choose? Why?*

Problem 5 *In this case, the king explained that, again, the signs were either both true or both false. Door 1: Either this room contains a tiger, or there is gold in the other room. Door 2: There is gold in the other room. Does the first room contain gold or a tiger? What about the other room?*

Problem 6 *A prince traveling through a magic land found an enchanted castle guarded by an evil goblin. The goblin told the prince that he had a box with a key to the castle gate, but that it was very dangerous to open the box. The prince immediately accepted the challenge. The goblin presented the young man with three boxes: red, blue, and green. The red box stated, "Here is the key." The blue box read, "The green box is empty." The green box warned, "There is a poisonous snake in this box." "Ha-ha-ha," laughed the goblin, "it is true that one of these boxes has the key, one is home to a deadly snake, and one is empty, but all the labels on the boxes lie. You can only try once. If you open the empty box you go home empty-handed, and if you open the box with the snake you die!" Help the prince to choose wisely.*

Problem 7 *A says, "I am a boy." B says, "I am a girl." One of them is a boy while the other is a girl. At least one of them is lying. Who is a boy and who is a girl?*

A Boolean algebra challenge

Problem 8 *Here is a famous puzzle known as the "zebra puzzle," said to have been invented by Albert Einstein as a child. The version below was published in Life International.*

- *There are five houses.*
- *The Englishman lives in the red house.*
- *The Spaniard owns the dog.*
- *Coffee is drunk in the green house.*
- *The Ukrainian drinks tea.*
- *The green house is immediately to the right of the ivory house.*
- *The Old Gold smoker owns snails.*
- *Kools are smoked in the yellow house.*
- *Milk is drunk in the middle house.*
- *The Norwegian lives in the first house.*
- *The man who smokes Chesterfields lives in the house next to the man with the fox.*
- *Kools are smoked in the house next to the house where the horse is kept.*
- *The Lucky Strike smoker drinks orange juice.*
- *The Japanese smokes Parliaments.*
- *The Norwegian lives next to the blue house.*

Each house is painted a different color and their inhabitants are of different nationalities, own different pets, drink different beverages, and smoke different cigarettes. Who drinks water? Who owns the zebra?

You may solve this problem any way you want, but one way to go about it is using Boolean algebra! Feel free to use the space on the next few pages to work on the problem.

House	1	2	3	4	5
Color					
Nationality					
Drink					
Smoke					
Pet					

If you have extra time...

A cryptarithmetic puzzle, also know as a cryptarithm or alphametic, is a math game of figuring out unknown numbers represented by words. Different letters correspond to different digits. Same letters correspond to same digits. The first digit of a number cannot be zero.

Problem 9 *Solve the following cryptarithm in German.*

$$\begin{array}{r} E\ I\ N\ S \\ E\ I\ N\ S \\ E\ I\ N\ S \\ +\ E\ I\ N\ S \\ \hline V\ I\ E\ R \end{array}$$