Week 3 Class Notes

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1 Review of Boolean Algebra Basics

1.1 Variables

Variables are assigned truth values (true and false) rather than numbers in elementary algebra. True and false are often denoted by 1 and 0, respectively.

1.2 Basic Operators

Similar to elementary algebra where addition, subtraction, multiplication, and division are the basic operations which define most of polynomial based mathematics, the basic operators of Boolean algebra are not, and, and or.

Operator	Notation	Definition
not	negation of A, A not, ¬A, A', $\overline{A}$	reverses the truth value
and	A and B, A ∧ B, A×B, AB	output is true if all compared expressions are true; output is false if at least one of the compared expression is false
or	A and B, A ∨ B, A+B	output is false if all compared expressions are false; output is true if at least one of the compared expression is true

1.2.1 NOT Example

A	$\neg A$
0	1
1	0

1.2.2 AND Examples

A	B	$A \times B$	A	B	C	$A \times B \times C$
0	0	0	0	$\times$	$\times$	0
0	1	0	$\times$	0	$\times$	0
1	0	0	$\times$	$\times$	0	0
1	1	1	1	1	1	1

1.2.3 OR Examples

A	B	$A + B$	A	B	C	$A + B + C$
0	0	0	0	0	0	0
0	1	1	1	$\times$	$\times$	1
1	0	1	$\times$	1	$\times$	1
1	1	1	$\times$	$\times$	1	1

1.3 Basic Laws and Theorems

Commutative Law

$$A + B = B + A \tag{1}$$

$$A \times B = B \times A \tag{2}$$

Associative Law

$$(A + B) + C = A + (B + C) \tag{3}$$

$$(A \times B) \times C = A \times (B \times C) \tag{4}$$

Distributive Law

$$A \times (B + C) = (A \times B) + (A \times C) \tag{5}$$

$$A + (B \times C) = (A + B) \times (A + C) \tag{6}$$

Identity Law

$$A + 0 = A \text{ and } A + 1 = 1 \tag{7}$$

$$A \times 0 = 0 \text{ and } A \times 1 = A \tag{8}$$

Idempotent Law

$$\underbrace{A + A + \ldots + A}_{\text{n times}} = A \tag{9}$$

$$\underbrace{A \times A \times \ldots \times A}_{\text{n times}} = A \tag{10}$$

Involution Law

$$\neg\neg A = A \tag{11}$$

Complement/Negation Law or Law of the Excluded Middle/Third

$$A + \neg A = 1 \tag{12}$$

$$A \times \neg A = 0 \tag{13}$$

De Morgan's Law

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \tag{14}$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \tag{15}$$

Law of Absorption or Domination Law

$$A + AB = A \tag{16}$$

$$A \times (A + B) = A \tag{17}$$

1.4 Walkthrough of Sample Problems

Problem 1. *Simplify the following:*

- $\neg(AB + C\neg D) + \neg B + \neg C$
- $A + A(\neg(B\neg C + \neg BD))$
- $X + XY + XYZ + X\neg YZ + XY\neg Z$

$$\begin{aligned} &\neg(AB + C\neg D) + \neg B + \neg C \\ &= (\neg A + \neg B) \times (\neg C + D) + \neg B + \neg C && \text{De Morgan's Law} \\ &= \neg A\neg C + \neg AD + \neg B\neg C + \neg BD + \neg B + \neg C && \text{Distributive Law} \\ &= \neg AD + \neg B\neg C + \neg BD + \neg B + \neg C && \text{Law of Absorption} \\ &= \neg AD + \neg B + \neg C && \text{Law of Absorption} \end{aligned}$$

$$\begin{aligned} &A + A(\neg(B\neg C + \neg BD)) \\ &= A && \text{Law of Absorption} \end{aligned}$$

$$\begin{aligned} &X + XY + XYZ + X\neg YZ + XY\neg Z \\ &= X && \text{Law of Absorption} \end{aligned}$$

2 Solving Logic Problems with Boolean Algebra

Logical problems can be solved using Boolean algebra. Generally, when solving these kinds of problems, the following steps can be followed to consistently arrive at the solution:

1. Assign variables to the simple statements of the problem.
2. Create Boolean expressions based on the information given in the problem.
3. Derive, combine, and simplify the Boolean expressions. Remove contradictions if needed. Remember the context of the problem to appropriately assign truth values to certain expressions.

2.1 Walkthrough of Sample Problems

Problem 2. *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many different possibilities do they have?*

Assign variables:

$$\begin{aligned} M_1 &= \text{First class is math class} \\ M_2 &= \text{Second class is math class} \\ H_1 &= \text{First class is history class} \\ H_3 &= \text{Third class is history class} \\ E_2 &= \text{Second class is English class} \\ E_3 &= \text{Third class is English class} \end{aligned}$$

Create the Boolean expressions:

$$\begin{aligned} T_M &= M_1 + M_2 = 1 \\ T_H &= H_1 + H_3 = 1 \\ T_E &= E_2 + E_3 = 1 \end{aligned}$$

Combine and simplify the expressions:

$$\begin{gathered} 1 = T_M T_H T_E \\ = (M_1 + M_2)(H_1 + H_3)(E_2 + E_3) \\ = (E_2)(M_1)(H_1) + (E_2)(M_1)(H_3) + (E_2)(M_2)(H_1) + (E_2)(M_2)(H_3) + \\ (E_3)(M_1)(H_1) + (E_3)(M_1)(H_3) + (E_3)(M_2)(H_1) + (E_3)(M_2)(H_3) \\ = (E_2)(0) + (E_2)(M_1)(H_3) + (0)(H_1) + (0)(H_3) + (E_3)(0) + (0)(M_1) + \\ (E_3)(M_2)(H_1) + (0)(M_2) \\ = (E_2)(M_1)(H_3) + (E_3)(M_2)(H_1) \end{gathered}$$

Therefore, there are two possible scenarios. The possible orders for the first, second and third classes are math, English, history or history, math, English.

Problems 5-7 are based on the following setup: Once upon a time in a land far, far away, there lived a king who invented the following way of punishing criminals. Convicted lawbreakers were given a choice between two doors. Behind each door, there could be either a hungry tiger or a treasure of gold, but not nothing or both. The king would also post some warnings on the doors and then let the criminals choose

Problem 3. *The king took the prisoner to the doors. There was a sign on each door. The first read, "There is gold in this room and there is a tiger in the other." The sign on the second door read, "There is gold in one of these rooms and in one of these rooms there is a tiger." "Are the signs true?" asked the prisoner. "One of them is," replied the king, "but the other is not. Now, make your choice, buddy!" Which door should the prisoner open? Why?*

Assign Variables:

$$G_1 = \text{First door has gold}$$
$$G_2 = \text{Second door has gold}$$
$$T_1 = \text{First door has a tiger}$$
$$T_2 = \text{Second door has a tiger}$$

Create the Boolean expressions:

$$S_1 = G_1 T_2$$
$$S_2 = (G_1 + G_2)(T_1 + T_2)$$

Note that:

$$S_1 \rightarrow S_2$$

So, $S_1 = 0$ since if $S_1 = 1$, then $S_1 S_2 = 1$. But we know that $S_1 S_2 = 0$. Therefore, there is a contradiction and $S_1 \neq 1$.

Therefore:

$$\begin{aligned} 1 &= \neg S_1 S_2 + S_1 \neg S_2 \\ &= \neg S_1 S_2 + 0 \neg S_2 \\ &= \neg S_1 S_2 \end{aligned}$$

From here, we can make logical assumptions and arrive at the answer that the gold is behind the second door and a tiger behind the other.

Problem 4. *For the second prisoner, the following signs were put on the doors. Door 1: at least one of these rooms contains gold. Door 2: a tiger is in the other room. "Are the signs true?" asked the prisoner. "They are either both true or both false," replied the king. Which door should the prisoner choose? Why?*

Create the Boolean expressions:

$$S_1 = G_1 + G_2$$
$$S_2 = T_1$$

Combine and simplify the expressions:

$$\begin{aligned} 1 &= S_1 S_2 + \neg S_1 \neg S_2 \\ &= (G_1 + G_2)(T_1) + \neg(G_1 + G_2)(\neg T_1) \\ &= (G_1 + G_2)(T_1) + (\neg G_1)(\neg G_2)(\neg T_1) \\ &= (G_1 + G_2)(T_1) + (0)(\neg G_2) \\ &= (G_1 + G_2)(T_1) \\ &= G_1 T_1 + G_2 T_1 \\ &= G_2 T_1 \end{aligned}$$

So, gold is behind door 2 and the tiger is behind door 1.

Problem 5. *In this case, the king explained that, again, the signs were either both true or both false. Sign 1: either this room contains a tiger, or there is gold in the other room. Sign 2: there is gold in the other room. Does the first room contain gold or a tiger? What about the other room?*

Create the Boolean expressions:

$$\begin{aligned} S_1 &= T_1 + G_2 \\ S_2 &= G_1 \end{aligned}$$

Combine and simplify the expressions:

$$\begin{aligned} 1 &= S_1 S_2 + \neg S_1 \neg S_2 \\ &= (T_1 + G_2)(G_1) + \neg(T_1 + G_2)(\neg G_1) \\ &= (T_1 + G_2)(G_1) + (\neg T_1)(\neg G_2)(\neg G_1) \\ &= (T_1 + G_2)(G_1) + (0)(\neg G_2) \\ &= (T_1 + G_2)(G_1) \\ &= T_1 G_1 + G_2 G_1 \\ &= G_2 G_1 \end{aligned}$$

There is gold behind both doors.

Problem 6. *A prince travelling through a magic land found an enchanted castle guarded by an evil goblin. The goblin told the prince that he had a box with a key to the castle gate, but that it was very dangerous to open the box. The prince immediately accepted the challenge. The goblin presented the young man with three boxes, red, blue, and green. It was written on the red box, "Here is the key." The blue box read, "The green box is empty." The green box had a warning, "There is a poisonous snake in this box." "Ha-ha-ha," laughed the goblin, "it is true that one of these boxes has the key, one is home to a deadly snake and one is empty, but all the labels on the boxes lie. You can only try once. If you open the empty box, you go home empty-handed and if you open the box with the snake, you die!" Help the prince to choose wisely.*

Assign variables:

$$\begin{aligned} R_K &= \text{Red box has the key} \\ G_E &= \text{Green box is empty} \\ G_P &= \text{Green box has the poisonous snake} \end{aligned}$$

Create, combine and simplify the Boolean expressions:

$$\begin{aligned} 1 &= \neg R_K \neg G_E \neg G_P \\ &= (R_E + R_P)(G_P + G_K)(G_K + G_E) \\ &= (R_E + R_P)(G_K + 0)(G_K + 0) \\ &= (R_E + R_P)(G_K)(G_K) \\ &= (R_E + R_P)(G_K) \end{aligned}$$

So, the green box has the key. However, we are unsure as to the status of the red and blue box.

Problem 7. *A says, "I am a boy". B says, "I am a girl". One of them is a boy while the other is a girl. At least one of them is lying. Who is a boy and who is a girl?*

Assign variables:

$$A_B = \text{A is a boy}$$
$$B_G = \text{B is a girl}$$

Create, combine and simplify the Boolean expressions:

$$\begin{aligned} 1 &= \neg A_B B_G + A_B \neg B_G + \neg A_B \neg B_G \\ &= 0 + 0 + \neg A_B \neg B_G \\ &= \neg A_B \neg B_G \end{aligned}$$

So, A is a girl while B is a boy.

Problem 8. *Here is a famous puzzle known as the "zebra puzzle," said to have been invented by Albert Einstein as a child. The version below was published in Life International.*

- *There are five houses.*
- *The Englishman lives in the red house.*
- *The Spaniard owns the dog.*
- *Coffee is drunk in the green house.*
- *The Ukrainian drinks tea.*
- *The green house is immediately to the right of the ivory house.*
- *The Old Gold smoker owns snails.*
- *Kools are smoked in the yellow house.*
- *Milk is drunk in the middle house.*
- *The Norwegian lives in the first house.*
- *The man who smokes Chesterfields lives in the house next to the man with the fox.*
- *Kools are smoked in the house next to the house where the horse is kept.*
- *The Lucky Strike smoker drinks orange juice.*
- *The Japanese smokes Parliaments.*
- *The Norwegian lives next to the blue house.*

Each house is painted a different color and their inhabitants are of different nationalities, own different pets, drink different beverages, and smoke different cigarettes. Who drinks water? Who owns the zebra?

This problem can be solved using a table such as the following:

House	1	2	3	4	5
Color	Yellow	Blue	Red	Ivory	Green
Nationality	Norwegian	Ukrainian	Englishman	Spaniard	Japanese
Drink	Water	Tea	Milk	Orange Juice	Coffee
Smoke	Kools	Chesterfield	Old Gold	Lucky Strike	Parliament
Pet	Fox	Horse	Snails	Dog	Zebra

So, the Norwegian drinks water and the Japanese owns the zebra.

Boolean algebra walkthroughs for this problem:
`https://code.energy/solving-zebra-puzzle/`
`https://www.researchgate.net/publication/279639366_Solving_zebra_puzzle_with_Boolean_algebra`