ORMC Intermediate I **July 9, 2023**

Oleg Gleizer
prof1140g@math.ucla.edu

Algebra of statements

A *statement* is an expression which is either *true* or *false*. For example, *Let's go!* is not a statement, while *My math teacher is not human!* is.

Problem 1 *In the space below, write two sentences that are statements in the above sense and two more that are not.*

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If a statement A is true, we write $A = 1$. If a statement A is false, we write $A = 0$.

Problem 2 *Determine which of the sentences below are statements. If a sentence is a statement, find its value.*

$A =$ 23 *is divisible by* 5.

$B =$ *Please don't smoke on board the aircraft.*

$C =$ 65 *is a real number.*

$D =$ *Wolfgang Mozart is a famous Russian hockey player.*

$E =$ *What time is it now?*

$F =$ *Get out of here!*

$G =$ *Math is fundamental for understanding all other sciences.*

If a statement mentions only one event, true or false, it is called *simple*. If a statement mentions more than one event, it is called *composite*. For example, the statement *I come to Math Circle by car* is simple, while the statement *I come to Math Circle by car or by bus* is composite.

Let A and B be statements. Let us define $A + B$ as the statement *A or B*. For example, if A = *Three is greater than two*, and B = *Three is greater than five*, then $A + B$ = *Three is greater than two or greater than five.*

The statement *A or B* is false if and only if both A and B are false. If either of the statements A or B is true, then *A or B* is true as well.

A	B	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

We can see from the above *truth table* that in the algebra of logic $0 + 0 = 0$, while $1 + 0 = 0 + 1 = 1 + 1 = 1$.

Problem 3 *Is logical addition commutative? Why or why not? Please write your explanation in the space below.*

Problem 4 *Prove that* $A + 0 = A$ *and* $A + 1 = 1$.

Give a verbal interpretation to the above algebraic statements.

Problem 5 *Prove that* $\underbrace{A + A + \ldots + A}_{n \text{ times}} = A$.

Problem 6 *Form the logical sum of the following three statements and find its value.*

$A =$ *The planet Earth revolves around the moon.*

$B =$ *The planet Earth revolves around Jupiter.*

$C =$ *The planet Earth revolves around the sun.*

$A + B + C =$

Problem 7 *Prove that in logical addition,* $(A + B) + C = A + (B + C)$ *by using the truth table below.*

A	B	C	$A+B$	$B+C$	$(A+B)+C$	$A+(B+C)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

We can also perform logical multiplication. Let us define $A \times B$ as the statement *A and B*. For example, if $A =$ *Three is greater than two*, and $B =$ *Three is greater than five*, then $A \times B =$ *Three is greater than two and greater than five*. Quite obviously, *A and B* is true if and only if both A and B are true.

A	B	$A \times B$
0	0	0
0	1	0
1	0	0
1	1	1

In the algebra of logic just like in the algebra of numbers, $0 \times 0 = 1 \times 0 = 0 \times 1 = 0$, while $1 \times 1 = 1$.

Problem 8 *Is logical multiplication commutative? Why or why not? Please write down your explanation in the space below.*

Problem 9 *Prove that* $A \times 0 = 0$ *and* $A \times 1 = A$.

Problem 10 *Prove that* $\underbrace{A \times A \times \ldots \times A}_{n \text{ times}} = A$.

Problem 11 *Prove that in logical multiplication,* $(A \times B) \times C = A \times (B \times C)$.

A	B	C	$A \times B$	$B \times C$	$(A \times B) \times C$	$A \times (B \times C)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

Problem 12 *Form the logical product of the following three statements and find its value.*

$A =$ *Lobsters live in the ocean.*

$B =$ *Mobsters do not live in the ocean.*

$C =$ *Lobsters are no better than mobsters.*

$A \times B \times C =$

Problem 13

$A =$ *Spiders have* 8 *legs.*

$B =$ *"To Kill a Mockingbird" is a hunter's guidebook.*

$C = 48$ *is divisible by* 12*.*

Form the following statements from the above A*,* B*, and* C *and find their value.*

$A \times (B + C) =$

$A \times B + A \times C =$

Problem 14 *Prove that*
$A \times (B + C) = (B + C) \times A = A \times B + A \times C.$

A statement A preceded by *It is not true that...* or a statement equivalent to such is called the *negation* of A and is denoted as $\neg A$. For example, the negation of the statement B from problem 13 reads as follows. *"To Kill a Mockingbird" is not a hunter's guidebook.* The following is the truth table for the negation.

A	$\neg A$
0	1
1	0

Problem 15 *In the space below, write down the negation of the statement: I come to Math Circle by car or by bus.*

Problem 16 *Write down your own composite statement and its negation.*

Statement:

Negation:

Problem 17 *Can the statement* $A + \neg A$ *be false? Why or why not?*

Write the fact down as an algebraic formula.

Problem 18 *Can the statement* $A \times \neg A$ *be true? Why or why not?*

Write the fact down as an algebraic formula.

Problem 19 *Find the following.*

$\neg\neg A =$

Problem 20

$A =$ *Bob is driving to work.*

$B =$ *Bob is playing the flute.*

$C =$ *Bob is eating a burger.*

Form the following statements.

- $AB + \neg C =$

- $(A + B)C =$

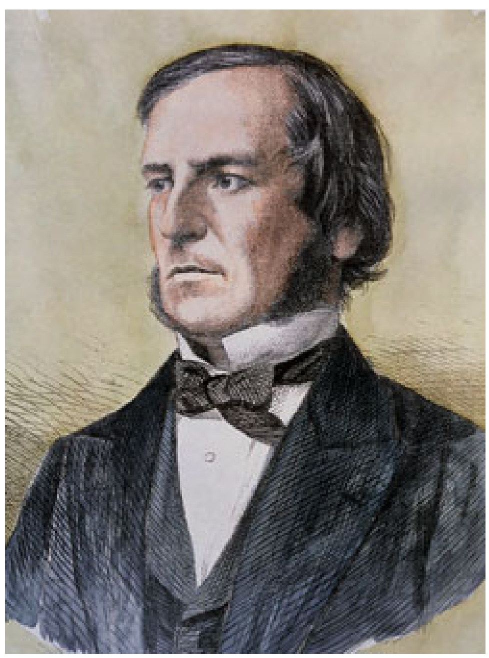
- $A\neg B + C =$

- $\neg A\neg BC =$

Problem 21 *Using the simple statements A, B, and C from problem 20, rewrite the following as a mathematical formula.*

It is not true that Bob is either driving to work and playing the flute or eating a burger.

The algebra of logic we have studied above is called *Boolean*, after George Boole (1815-1864), an English mathematician, philosopher and logician.



George Boole

Below you will find one more feature of Boolean algebra that truly distinguishes it from every other algebraic structure you have seen before.

In problem 14, we have proven that, similar to the algebra of numbers, multiplication in Boolean algebra is distributive.

$$A \times (B + C) = A \times B + A \times C$$

In Boolean algebra, unlike the algebra of numbers, addition is distributive with respect to multiplication as well!

$$A + (B \times C) = (A + B) \times (A + C) \tag{1}$$

Problem 22 *Prove formula 1.*

A	B	C	$B \times C$	$A + (B \times C)$	$A + B$	$A + C$	$(A + B) \times (A + C)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Negation of composite statements

The two formulas proven in problems 23 and 25 below are fundamental for understanding the algebra of logic.

Problem 23 *Prove that* $\neg(A + B) = \neg A \times \neg B$.

A	B	$A + B$	$\neg(A + B)$	$\neg A$	$\neg B$	$\neg A \times \neg B$
0	0					
0	1					
1	0					
1	1					

Problem 24 *Negate the statement: My dad likes to watch football or baseball.*

Problem 25 *Prove that* $\neg(A \times B) = \neg A + \neg B$.

A	B	$A \times B$	$\neg(A \times B)$	$\neg A$	$\neg B$	$\neg A + \neg B$
0	0					
0	1					
1	0					
1	1					

Problem 26 *Negate the statement: My dad likes to watch football and baseball.*

The following two formulas generalize the ones proven in problems 23 and 25. They are referred to as De Morgan's Laws.

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \qquad (2)$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \qquad (3)$$

Simplifying Boolean expressions

Two expressions of Boolean algebra are called *equivalent* if they are equal as functions; i.e., the same inputs produce the same outputs. We can check this using the corresponding truth tables. For example, the expressions $A+BC$ and $(A+B)(A+C)$ are equivalent.

Problem 27 *Prove that the expressions* $A + AB$ *and* A *are equivalent without using a truth table.*

Then use the truth table to check your proof.

A	B	AB	$A + AB$
0	0		
0	1		
1	0		
1	1		

In problem 27, we proved the following remarkable Boolean algebra equivalence.

$$A + AB = A \tag{4}$$

Here is one more.

Problem 28 *Prove the following formula without using a truth table.*

$$A(A + B) = A \tag{5}$$

To *simplify* a Boolean algebra expression is to find an equivalent expression that

1. contains no negations of composite statements, and
2. has as few simple statements as possible.

The equivalence

$$A(A + \neg(VW + \neg XYZ)) = A \tag{6}$$

is an example of such a simplification.

Problem 29 *If you were to check formula 6 using a truth table, how many different inputs (variables) would you need to consider?*

Problem 30 *Prove formula 6.*

The following two problems present two more very important equivalences.

Problem 31 *Prove the following formula.*

$$AB + A\neg B = A \tag{7}$$

Problem 32 *Prove the following formula.*

$$(A + B)(A + \neg B) = A \tag{8}$$

Problem 33 *Simplify the following Boolean algebra expressions.*

- $X = \neg(AB) + \neg B$

  $X =$

- $Y = \neg(\neg BC + C)$

  $Y =$

- $Z = \neg(\neg AC) + B\neg C$

  $Z =$

Problem 34

A = *Bob is driving to work.*

B = *Bob is playing the flute.*

Form the statement $X = \neg(AB) + \neg B$ *from problem 33 in plain English and simplify it in the space below.*

Problem 35 *Simplify the following Boolean algebra expressions.*

- $\neg(AB) + \neg A =$

- $A + \neg(\neg AB) =$

- $\neg(\neg A \neg B) + \neg A =$

- $\neg(A + \neg(\neg AB)) =$

- $\neg(AB) + \neg ABC =$

- $\neg A + \neg(AB + \neg B) =$

- $A \neg BC + A \neg(BC) + ABC + A \neg B =$

The problem below is similar to the logical problems you have solved at the Circle and/or at various math competitions. Please solve it any way you like. Afterward, we will show you how to solve the problem using Boolean algebra.

Problem 36 *The year is 3014. Four kids got to the final tour of GMC8 (Galactic Math Olympiad for 8th graders): Nathan, Michelle, Laura, and Reinhardt. Some knowledgeable ORMC fans discussed their chances to win. One student thought that Nathan would take first place and Michelle would take second. Another student thought that Laura would take second place while Reinhardt would end up the last of the four. The third student thought that Nathan would be second and Reinhardt third. When the results of the competition came out, it turned out that each of the ORMC students had made only one of their two predictions correctly. Find the places Nathan, Michelle, Laura, and Reinhardt got at GMC8-3014.*

A Boolean algebra solution to problem 36

The following are the simple statements from problem 36.

- $N_1 =$ *Nathan takes the first place.*
- $M_2 =$ *Michelle takes the second place.*
- $L_2 =$ *Laura takes the second place.*
- $R_4 =$ *Reinhardt takes the fourth place.*
- $N_2 =$ *Nathan takes the second place.*
- $R_3 =$ *Reinhardt takes the third place.*

Let us use the simple statements above to translate the story into the Boolean algebra language. The first fan made a composite statement N_1M_2 that turned out to be false.

$$N_1M_2 = 0$$

The fact that a half of the guess is true means that either $N_1\neg M_2 = 1$ and $\neg N_1M_2 = 0$ or that $N_1\neg M_2 = 0$ and $\neg N_1M_2 = 1$. This can be expressed by means of a single formula.

$$N_1\neg M_2 + \neg N_1M_2 = 1 \tag{9}$$

A similar translation of the other two fans' predictions into the Boolean algebra language gives us the following.

$$L_2\neg R_4 + \neg L_2R_4 = 1 \tag{10}$$

$$N_2\neg R_3 + \neg N_2 R_3 = 1 \tag{11}$$

Multiplying 9, 10, and 11 brings together all the information we have about the competition.

$$(N_1\neg M_2+\neg N_1 M_2)(L_2\neg R_4+\neg L_2 R_4)(N_2\neg R_3+\neg N_2 R_3) = 1 \tag{12}$$

Let us first find the product of the second and third factors.

$$(L_2\neg R_4 + \neg L_2 R_4)(N_2\neg R_3 + \neg N_2 R_3) = 1$$

Opening parentheses gives the following.

$$L_2\neg R_4 N_2\neg R_3+L_2\neg R_4\neg N_2 R_3+\neg L_2 R_4 N_2\neg R_3+\neg L_2 R_4\neg N_2 R_3 = 1$$

Since Laura and Nathan cannot take the second place simultaneously, $L_2\neg R_4 N_2\neg R_3 = 0$. Since Reinhardt cannot take the third and fourth place at the same time, $\neg L_2 R_4\neg N_2 R_3 = 0$. The above sum shortens to just two terms.

$$L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3 = 1$$

This way, 12 boils down to the following.

$$(N_1\neg M_2 + \neg N_1 M_2)(L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3) = 1$$

Let us expand. $N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 + N_1\neg M_2\neg L_2 R_4 N_2\neg R_3 + \neg N_1 M_2 L_2\neg R_4\neg N_2 R_3 + \neg N_1 M_2\neg L_2 R_4 N_2\neg R_3 = 1$ Since $N_1 N_2 = 0$, the second term is equal to zero. Since $M_2 L_2 = 0$, the third term is equal to zero as well. Since $M_2 N_2 = 0$, the same is true for the last term. We end up with the equation

$$N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 = 1$$

that tells us the results of the competition. Nathan takes the first place, Laura the second, Reinhardt the third. Therefore, Michelle takes the fourth place. There are no contradictions: Michelle is not second, Reinhardt is not fourth, and Nathan is not second. We have solved the problem!

Homework

Problem 37 *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many possible schedules are there?*

Problem 38 *Consider the sentence: This statement is false. Is this a statement? If you think it is, find its value. If you think it is not, give a reason.*