Nonstandard analysis

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The conduit is available at `https://tinyurl.com/ORMCconduit`.

1 Dual numbers

Definition 1. We define a *dual number* to be an expression of the form $a + b\varepsilon$, where $\varepsilon^2 = 0$ (just as complex numbers are expressions of the form $a + bi$, where $i^2 = -1$).

In other words, the rules for addition, subtraction, and multiplication are as follows:

- $(a + b\varepsilon) + (c + d\varepsilon) = (a + c) + (b + d)\varepsilon$;
- $(a + b\varepsilon) - (c + d\varepsilon) = (a - c) + (b - d)\varepsilon$;
- $(a + b\varepsilon) \cdot (c + d\varepsilon) = ac + (ad + bc)\varepsilon$.

Definition 2. If $f(x)$ is an algebraic function (everything with powers and roots), then we write $f(x + \varepsilon) = f(x) + f'(x)\varepsilon$. All functions in this worksheet will be algebraic. The coefficient $f'(x)$ is called the *derivative* of f. If $f(x+\varepsilon)$ is undefined, we say that f is not differentiable at x.

Example 1. The function $f(x) = x^2$ has a derivative $f'(x) = 2x$, because $f(x + \varepsilon) = (x + \varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2 = f(x) + (2x)\varepsilon$.

Remark 1. Also, for algebraic functions $f(x + k\varepsilon) = f(x) + kf'(x)\varepsilon$.

Problem 1. Find the derivative of the function $f(x) = x^n$.

Problem 2 (✐). If $f(x)$ and $g(x)$ are two functions with known derivatives $f'(x)$ and $g'(x)$, find the derivatives of $f(x) + g(x)$ and $f(x)g(x)$.

Problem 3 (✐). a) When can you divide dual numbers? I.e. for which dual numbers $(a + b\varepsilon)$ there is a dual number $(x + y\varepsilon)$ such that $(a + b\varepsilon) \cdot (x + y\varepsilon) = 1$?

b) Find an explicit formula for the inverse and use it to find the derivative of $f(x) = \frac{1}{x}$.

Problem 4. a) Which dual numbers have a square root? I.e. for which dual numbers $(a + b\varepsilon)$ there is a dual number $(x + y\varepsilon)$ such that $(x + y\varepsilon)^2 = a + b\varepsilon$?

b) Find an explicit formula for the square root and use it to find the derivative of $f(x) = \sqrt{x}$.

Problem 5 (✎). Find the derivatives of functions

a) $f(x) = \frac{x}{1+x^2}$,

b) $g(x) = \sqrt{1-x^2}$.

Problem 6. If $f(x)$ and $g(x)$ are two functions with known derivatives $f'(x)$ and $g'(x)$, find the derivative of $f(g(x))$.

Problem 7 (*). The *Bring radical* $\mathrm{BR}(a)$ of a real number a is defined as the only solution to the equation $x^5 + x + a = 0$. In dual numbers, find a solution to the equation $x^5 + x + (a + b\varepsilon) = 0$ (your answer will include $\mathrm{BR}(a)$ and it's OK). Use this solution to find the derivative of the Bring radical.

Definition 3. The *inverse function* of a function f is the function g such that $f(x) = y$ is equivalent to $g(y) = x$. So if $f(x) = x^3$, $g(x) = x^{\frac{1}{3}}$. The inverse is denoted as f^{-1}.

Problem 8 (*). If $f(x)$ is a function with a known derivative $f'(x)$ and an inverse $f^{-1}(x)$, find the derivative of $f^{-1}(x)$. Use the formula you obtained to rederive the derivatives of $\sqrt{x}$ and $\mathrm{BR}(x)$.

Remark 2. Dual numbers provide a reliable way to differentiate in computers: see `https://tinyurl.com/wikidualnumbers`

2 Nonarchimedean extensions

Definition 4. An *ordered field* is a set F of elements, with defined operations of addition $x + y$, multiplication $x \cdot y$ and comparison $x < y$ such that:

1. $(x + y) + z = x + (y + z)$;
2. There is an element 0 such that $x + 0 = x$;
3. For every x there is a negative $-x$ such that $x + (-x) = 0$;
4. $x + y = y + x$;
5. $(x + y)z = xz + yz$;
6. $(xy)z = x(yz)$;
7. There is an element 1 such that $x \cdot 1 = x$;
8. For every x except 0, there is an inverse $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$;
9. $xy = yx$;
10. The comparison $x < x$ is always false;

11. If $x < y$ and $y < z$, then also $x < z$;
12. If $x < y$ then $x + z < y + z$;
13. If $x < y$ and $z > 0$, then $xz < yz$;
14. $0 < 1$;
15. Either $x < y$ or $y < x$ or $x = y$.

We will work with ordered fields that contain $\mathbb{R}$. We call these *nonarchimedean extensions* of $\mathbb{R}$.

Proposition 1. *In any ordered field for any x except 0 we have $\frac{1}{\frac{1}{x}} = x$*

Proof.

1. By axiom 8 applied to $\frac{1}{x}$ we have $\frac{1}{x} \cdot \frac{1}{\frac{1}{x}} = 1$;
2. Multiplying both parts of the previous formula by x we get $x \cdot (\frac{1}{x} \cdot \frac{1}{\frac{1}{x}}) = x \cdot 1$;
3. By axiom 6 rewrite the LHS: $(x \cdot \frac{1}{x}) \cdot \frac{1}{\frac{1}{x}} = x \cdot 1$;
4. By axiom 8 we can change $x \cdot \frac{1}{x}$ in the LHS to 1: $1 \cdot \frac{1}{\frac{1}{x}} = x \cdot 1$;
5. By axiom 9 applied to x and 1 we rewrite the LHS again: $\frac{1}{\frac{1}{x}} \cdot 1 = x \cdot 1$;
6. Finally, by axiom 7 applied in both parts, $\frac{1}{\frac{1}{x}} = x$.

□

Problem 9 (✎). Prove in the same fashion:

a) $0 \cdot x = 0$;

b) $(-x)(-y) = xy$;

c) If $0 < x < y$, then $\frac{1}{x} > \frac{1}{y}$;

d) Let $|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$

Prove that $|x| + |y| \geq |x + y|$.

Definition 5. An element $\delta \in F$ is called *infinitesimal* if $|n\delta| < 1$ (i.e $-1 < n\delta < 1$) for all natural numbers n. An element $x \in F$ is called *limited*, if $|x| < n$ for some natural number n, and *unlimited* otherwise. We denote infinitesimal elements by δ, ϵ, and unlimited elements by H, K (from the word Huge).

Problem 10. Prove that a positive element δ is infinitesimal if and only if it is smaller than every positive real. Analogously, a negative element is infinitesimal if and only if it is bigger than every negative real.

Problem 11 (✎). Prove the following properties:

a) If δ, ε are infinitesimal, the $\delta + \varepsilon$ is infinitesimal;

b) If δ is infinitesimal, and a is limited, then $a\delta$ is infinitesimal;

c) If a and b are limited, then so are ab and $a + b$.

d) A nonzero element δ is infinitesimal if and only if an element $\frac{1}{\delta}$ is unlimited.

Problem 12. Let δ be a positive infinitesimal. Which of the following is greater:

a) δ or δ^2?

b) $1 - \delta$ or $\frac{1}{1+\delta^2}$?

c) $\frac{1+\delta}{1+\delta^2}$ or $\frac{2+\delta^2}{2+\delta^3}$?

Recall that if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial, then its *leading term* is $a_n x^n$, and the *leading coefficient* is a_n.

Problem 13 (*)**.** Consider the set $\mathbb{R}(x)$ of all rational functions $\frac{P(x)}{Q(x)}$ with usual arithmetic operations. We define an order by $\frac{P(x)}{Q(x)} > \frac{R(x)}{S(x)}$ if the leading coefficients of $P(x)S(x) - Q(x)R(x)$ and $Q(x)S(x)$ have the same sign.

a) Prove that $\mathbb{R}(x)$ is a nonarchimedean extension of $\mathbb{R}$.

b) Prove that x is unlimited.

Problem 14 (*)**.** Let F be a nonarchimedean extension of $\mathbb{R}$ and let x be a positive unlimited element. Prove that $\frac{P(x)}{Q(x)} > \frac{R(x)}{S(x)}$ if and only if the condition from the previous problem holds. So there is a unique way to extend $\mathbb{R}$ by a single unlimited element.

Definition 6. Two elements x and y are said to be *infinitely close*, denoted $x \sim y$ if their difference $x - y$ is infinitesimal. A *real* number x_0 is said to be the *standard part* of x, denoted $\mathrm{st}(x)$, if it is infinitely close to x.

Problem 15 (✎)**.** Let H be positive unlimited. Find which of the following elements are limited, and find their standard parts:

$$\frac{1}{H},\ H^2 - 10H - 5,\ \frac{H+1}{H+2},\ \frac{H^3}{(H+1)^2},\ \sqrt{H+1} - \sqrt{H-1} \text{ (assume both square roots exist).}$$

Problem 16 (✎)**.** Prove that any limited element has a unique standard part. *Hint: for existence, consider* $\sup\{a \in \mathbb{R}, a < x\}$.

Problem 17. Prove that $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$ and $\mathrm{st}(xy) = \mathrm{st}(x)\mathrm{st}(y)$.