ORMC Intermediate I **June 25, 2023**

Oleg Gleizer
prof1140g@math.ucla.edu

Warm-up

Problem 1 *A student was asked to divide some number by two and to add three to the result. By mistake, the student multiplied the number by two and subtracted three from the product. Accidentally, the student got the right answer. Find the original number.*

Problem 2 *The Celsius temperature scale divides the temperature segment where the (distilled) water stays liquid into 100 equal parts. At the sea level,* 0° *is the freezing point,* $100^\circ C$ *is the boiling one. In the Fahrenheit scale, the (distilled) water freezes at* $32^\circ F$ *and boils at* $212^\circ F$*. The transition between the scales is given by the following formula.*

$$T^\circ F = a \times T^\circ C + b$$

- *Find* a *and* b*.*

- *What temperature is the same in both scales?*

Problem 3 *Alice bought some chocolates and lollipops in the ratio 7:4. The price of all the chocolates to the price of all the lollipops was in the ratio 5:2. The girl spent $127.40 in all. If each lollipop costs 15 cents less than each chocolate, how many lollipops did Alice buy?*

Binary numbers

Let us recall that there are only two digits in the *binary system*, 0 and 1. $0_{10} = 0_2$ (remember, the subscript denotes the base), $1_{10} = 1_2$, but $2_{10} = 10_2$, $3_{10} = 11_2$, and so on.

Example 1 *Find the binary representation of the number* 174_{10}.

Let us list all the powers of 2 *that are less than or equal to* 174.

n	2^n
0	1
1	2
2	4
3	8
4	16

n	2^n
5	32
6	64
7	128
8	256

It turns out that the largest integral power of 2 *still less than*

174 is $128 = 2^7$.

$$174 = 128 + 46 = 2^7 + 46$$

The largest power of two less than 46 *is* $32 = 2^5$.

$$174 = 128 + 32 + 14 = 2^7 + 2^5 + 14$$

Finally, it is not hard to represent 14 *as a sum of powers of two,* $14 = 8 + 4 + 2$.

$$174 = 128 + 32 + 8 + 4 + 2 = 2^7 + 2^5 + 2^3 + 2^2 + 2^1$$

To write the number 174_{10} *in the binary form, we now need to fill the following eight boxes with either zeros or ones.*

⊔	⊔	⊔	⊔	⊔	⊔	⊔	⊔
7	6	5	4	3	2	1	0

The numbers under the boxes are the powers of two. If a power is absent from the decomposition of the number, then the corresponding box is filled with zero. For example, there is no $1 = 2^0$ *in the decomposition of the number* 174 *we have computed, so the first box from the right is filled with zero.*

							0
7	6	5	4	3	2	1	0

$2 = 2^1$ *is present in the decomposition, so the box corresponding to the first power gets filled with one.*

						1	0
7	6	5	4	3	2	1	0

$4 = 2^2$ *is also present in the decomposition, so the box corresponding to the second power of two gets filled with one.*

					1	1	0
7	6	5	4	3	2	1	0

Filling up all the boxes gives us the binary representation.

1	0	1	0	1	1	1	0
7	6	5	4	3	2	1	0

We write it down as follows.

$$174_{10} = 10101110_2$$

Problem 4 *Find the binary representations of the following decimal numbers.*

$$12_{10} =$$

$$25_{10} =$$

$$32_{10} =$$

$$100_{10} =$$

Problem 5 *Find the decimal representations of the following binary numbers.*

$$101_2 =$$

$$11001_2 =$$

$$1000000_2 =$$

$$1010011_2 =$$

Problem 6 *Complete the binary addition and multiplication tables below.*

+	0	1
0		
1		

×	0	1
0		
1		

Problem 7 *Use long addition to sum up the following two binary numbers without switching to the decimals.*

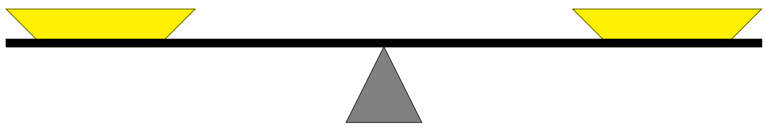
$$\begin{array}{rcccccc} & 1 & 1 & 0 & 1 & 1 & 1 \\ + & & 1 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

Then find the decimal representation of the summands and of the sum and check your answer.

$$110111_2 =$$

$$10011_2 =$$

A balance scale is a device for comparing weights very similar to a see-saw at a children's playground. You put two weights on the scales' plates. If the weights are equal, the scales remain in balance. If the weights are different, the lighter weight goes up.



Problem 8 *Given a balance scale and the weights of 1 lb, 2 lbs, 4 lbs, and 8 lbs (one of each), prove that you can weigh any (integral) load from 1 to 15 lbs. Why do you think this is possible?*

Problem 9 *Perform the following long multiplication without switching to the decimals.*

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1 \\ \times \quad 1\ 0\ 1\ 0 \\ \hline \end{array}$$

Then find the decimal representations of the factors and of the product and check your answer.

$$11011_2 =$$

$$1010_2 =$$

Problem 10 *Find the missing binary digits.*

$$
\begin{array}{cccccc}
 & & 1 & \sqcup & 0 & 1 \\
+ & & & \sqcup & \sqcup & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 0
\end{array}
\qquad\qquad
\begin{array}{cccccc}
 & & 1 & \sqcup & 0 & 1 \\
+ & & & 1 & 0 & \sqcup \\
\hline
 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

$$
\begin{array}{ccccc}
 & & 1 & \sqcup & 1 \\
\times & & & & \sqcup \\
\hline
 & & 1 & 0 & 1
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & & 1 & \sqcup \\
\times & & & \sqcup & 1 \\
\hline
 & 1 & 0 & 0 & 1
\end{array}
$$

Problem 11 *Recover the missing binary digits making the below inequalities correct.*

$$1\ 0\ 0\ \sqcup\ \sqcup \quad > \quad 1\ 0\ 0\ 1\ 0$$

$$1\ 0\ \sqcup\ \sqcup\ 0 \quad > \quad 1\ 0\ 1\ 0\ 0$$

$$1\ \sqcup\ 0\ 1\ \sqcup \quad > \quad 1\ 1\ 0\ 1\ 0$$

Problem 12 *Perform the following subtraction of the binary numbers.*

$$\begin{array}{rccccc} & 1 & 0 & 0 & 1 & 0 \\ - & & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

Then find the decimal representations of the numbers and of the difference and check your answer.

$$10010_2 =$$

$$1011_2 =$$

Problem 13 *Solve the following equations in the binaries.*

$$x + 11 = 1101 \qquad\qquad x =$$

$$x - 10 = 101 \qquad\qquad x =$$

$$x - 1101 = 11011 \qquad\qquad x =$$

$$x + 1110 = 10001 \qquad\qquad x =$$

$$x + 111 = 11110 \qquad\qquad x =$$

$$x - 1001 = 11101 \qquad\qquad x =$$

Complement of a decimal number

A *decimal complement* of a positive decimal integer x is a positive decimal integer c such that $x + c$ equals the smallest number of the form 10^n greater than or equal to x. For example, the decimal complement of 7 is 3 $(7 + 3 = 10)$.

Let us find the decimal complement of 243. The smallest power of ten greater than the number is 1000. To have zero as the last digit of the sum, the last digit of the complement must be equal to 7.

$$c = \underline{\quad}\ \underline{\quad}\ \underline{\ 7\ }$$

$3 + 7 = 10$, so one carries over. Therefore, the second digit of the complement must be equal to five.

$$c = \underline{\quad}\ \underline{\ 5\ }\ \underline{\ 7\ }$$

$1 + 4 + 5 = 10$, so one carries over again. Hence, the third digit of c must be equal to seven.

$$c = \underline{\ 7\ }\ \underline{\ 5\ }\ \underline{\ 7\ }$$

The number c is the desired complement.

$$243 + 757 = 1000$$

Problem 14 *Find decimal complements of the following decimal numbers.*

x	c
24	
179	
$7,834$	

A decimal complement is a great computational tool that replaces a trickier operation, subtraction, with an easier one, addition. For example, let us see how it helps us to subtract $7,834$ from $9,632$.

$$9,632-7,834 = 9,632-(10,000-2,166) = 9,632+2,166-10,000$$

Instead of subtracting $7,834$ from $9,632$, we add the decimal complement of $7,834$, the number $2,166$, to $9,632$ and then subtract $10,000$.

$$9,632-7,834 = 9,632+2,166-10,000 = 11,798-10,000 = 1,798$$

Problem 15 *Use the above trick to solve the following subtraction problems.*

$$92 - 24 =$$

$$533 - 179 =$$

$$1025 - 787 =$$

Complement of a binary number

A *binary complement* of a positive binary integer x is a positive binary integer c such that $x + c$ equals the smallest number of the form 2^n greater than or equal to x. For example, the decimal complement of 101 is 11 ($101 + 11 = 1000$). We can use a method known as *two's complement* to construct c from x. The following are the steps involved in the two's complement.

1. Invert all the digits of the binary number.

2. Take the inverted binary number and add 1.

Let us use the algorithm to find the binary complement of the number $x = 10101100$.

The algorithm tells us to start by inverting all the digits. In other words, it tells us to change the 0s to 1s and 1s to 0s.

$$10101100 \rightarrow 01010011$$

Next, we add 1 to the inverted binary number.

$$\begin{array}{cccccccccc} & & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ + & & & & & & & & & 1 \\ \hline & & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}$$

Dropping the zero at the front of the complement, we get the following number.

$$c = 1010100$$

Problem 16 *Use long addition to check that c is indeed the complement of x.*

$$
\begin{array}{r}
1\ 0\ 1\ 0\ 1\ 1\ 0\ 0 \\
+ \quad 1\ 0\ 1\ 0\ 1\ 0\ 0 \\
\hline
\end{array}
$$

What power of two do x and c add up to?

$n =$

Problem 17 *Find binary complements of the following numbers.*

$x = 111$ $\qquad\qquad$ $c =$

$x = 1100$ $\qquad\qquad$ $c =$

Similar to the decimal case, we can use a binary complement to make the operation of subtraction simpler. Suppose that we need to subtract a binary number x from a binary number y. Since the sum of x and its complement c equals some power of two,

$$x + y = 2^n = 1\,\underbrace{0\ldots0}_{n\ \text{times}}\,{}_2,$$

we can perform the subtraction as follows.

$$y - x = y - (2^n - c) = y + c - 2^n$$

This way we replace a general subtraction problem by an addition problem followed by a subtraction of the simplest possible kind, that of a number having a one for the first digit and zeros for the rest of them.

To show the efficiency of the method, let us use it to solve Problem 12. We need to subtract 1011 from 10010 as shown below.

$$\begin{array}{rccccc} & 1 & 0 & 0 & 1 & 0 \\ - & & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

Let us rewrite the smaller number first. Instead of 1011, let us write its binary complement we are about to add. The complement is found using the algorithm on page 17. Finally, let us write down the sum of 1011 and its complement that we will subtract.

$$\begin{array}{r} 1\;0\;0\;1\;0 \\ +\quad 0\;1\;0\;1 \\ -\;1\;0\;0\;0\;0 \\ \hline \end{array}$$

Performing first the addition and then the subtraction completes the computation.

$$\begin{array}{r} 1\;0\;1\;1\;1 \\ -\;1\;0\;0\;0\;0 \\ \hline 1\;1\;1 \end{array}$$

Question 1 *It may be that the new subtraction algorithm is a bit easier than the old one, but do we really need it? And what is the point in turning subtracting problems into adding ones?*

We don't, but our computers do. The heart of the computer is a chip known as a CPU, central processing unit. Nearly all it can do is add binary numbers. So by turning any subtraction problem into an addition one, you just have learned how it does subtraction!

Problem 18 *Use the two's complement method to solve the following subtraction problems.*

$$\begin{array}{r} 1\,0\,1\,0\,1 \\ -\quad\quad 1\,1\,0 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 1\,1\,1\,0\,1 \\ -\quad 1\,0\,1\,1 \\ \hline \end{array}$$

Signed binary numbers

So far, we have been working with unsigned binary integers, which represent only positive values.

Signed binary numbers are a way to represent both negative and positive numbers by only using bits. They are necessary in computers since RAM and CPU registers recognize numbers as a sequence of bits without any special symbols such as the minus sign (-).

Signed binary numbers are calculated similarly to their unsigned counterparts, with the primary distinction lying in the treatment of the most significant (left-most) bit; the value is subtracted from rather than added to the resulting number.

It is important to note that there are multiple ways to represent signed binary integers. But for the purposes of this class, we will focus on learning the most commonly used representation.

Either way, let us calculate $x = 1110_2$ using signed values.

$$1(-2^3) + 1(2^2) + 1(2^1) + 0(2^0) = -8 + 4 + 2 + 0 = -2$$

Problem 19 *Find the signed value representations of the binary numbers below.*

$$1011_2 =$$

$$01011_2 =$$

Question 2 *In problem 19, why does the extra leading zero bit significantly impact the resulting value? Discuss the relationship between the number of bits used to represent a number and the range of values that can be expressed within the signed binary system.*

In computer systems, numbers are analyzed using a fixed number of bits, such as 4 or 8 bits. This limitation defines the range of values that can be represented. For example, below is a table showing the unsigned and signed values for 3-bit integers.

Binary Number	**Unsigned Value**	**Signed Value**
000	0	0
001	1	1
010	2	2
011	3	3
100	4	-4
101	5	-3
110	6	-2
111	7	-1

Question 3 *How does the signed value of a binary integer differ from that of its unsigned counterpart?*

Problem 20 *Fill in the corresponding values of 4-bit signed integers in the table below.*

Binary Number	**Signed Value**
0000	
0001	
0010	
0011	
0100	
0101	
0110	
0111	
1000	
1001	
1010	
1011	
1100	
1101	
1110	
1111	

Question 4 *Explore the relationship between the two's complement and signed values. Consider how one can be derived from the other.*

Basic arithmetic with signed integers

Under signed numbers, we can derive a negative number, c, by taking the equivalent positive value, x, and by carrying out the process of two's complement on it. Note that the negative counterpart of its positive value is just the binary complement. Having the negative number be the binary complement is not only an efficient method, but also simplifies arithmetic operations and logic circuitry.

Let us explore addition and subtraction with signed binary numbers.

Adding numbers with positive signed integers is the same as with addition with unsigned integers.

As for subtraction, recall that most computer systems can only add. So, instead of directly subtracting a number, you add its negative counterpart, or, as we established, its binary complement.

Further, keep in mind that computer systems operate on a fixed number of bits. So, the minuend, subtrahend, and difference should have the same number bits, and any overflow should be dealt with accordingly. An overflow occurs when the resulting number exceeds the representable range for a given number of bits.

Example 2 *Solve* $7 - 7$ *using signed binary numbers.*

$$7_{10} - 7_{10} = 7_{10} + (-7_{10}) = 0111_2 + 1001_2 = 10000_2 \rightarrow 0000_2$$

Problem 21 *Solve the following problems using signed binary numbers. Use the two's complement when necessary.*

$$57 - 61 =$$

$$42 + 9 =$$

$$33 - 28 =$$

Problem 22 *Perform the subtraction with the following unsigned binary numbers by taking the two's complement of the subtrahend.*

$$11010 - 10001 =$$

$$101001 - 101 =$$

If you are finished doing all the above, but there still remains some time...

Problem 23 *Three farmers went to the market to sell chickens. One farmer brought 10 chickens to sell, the second brought 16, while the third brought 26. The farmers agreed to charge the same price for their animals. After the lunch break, afraid that they may not sell all of the chickens, the farmers lowered the price. As a result, they sold all the chickens, making \$35 each. What was the price before and after the lunch break?*

Homework

An *algorithm* is a clear finite set of instructions needed to perform a computation or to solve a problem. For example, the algorithm you have seen on page 17 prescribes the steps needed to efficiently execute the subtraction of binary numbers.

Problem 24 *Design an algorithm for the division of binary numbers. Hint: realize division as repeated subtraction.*