Real numbers

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For the purposes of this worksheet, real numbers are defined as decimals, finite or infinite.

Problem 1 (✎). Represent $2.3\overline{17}$ (it's a shorthand for $2.3171717171717\ldots$) as a simple fraction.

Problem 2. Represent $\frac{2}{11}$ as an infinite decimal and prove that your representation works.

Problem 3. Show that $0.\overline{9}\ldots = 1$.

Remark 1. The same will be true even in nonstandard analysis. There is no real number 0.(0)1 with digit 1 "at infinity". So some numbers have two decimal representations, while "most" of them have only one.

Problem 4 (✎). Prove that the number $0.123456789101112131415\ldots$ obtained by writing all natural numbers in increasing order is irrational.

Problem 5 (✎). Prove that between any two different real numbers there exists a rational number.

Problem 6 (*). Prove that every real number from $[0, 1]$ can be represented as a sum of 9 numbers whose decimal representations contain only digits 0 or 8.

Problem 7 (*). Two genies take an infinite amount of turns and write the digits of an infinite decimal. The first genie, on every turn, writes any finite amount of digits to the tail of the decimal. The second genie writes one digit to the end. If the resulting decimal after an infinite amount of turns is periodic, the first genie wins; otherwise, the second genie wins. Who has a winning strategy?

Definition 1. Let M be a subset of the set of real numbers $\mathbb{R}$. A number $c \in \mathbb{R}$ is called an upper bound of the set M if $c \geq m$ for all $m \in M$. A number $c \in \mathbb{R}$ is called the least upper bound (or supremum) of the set M if c is an upper bound of M, but no smaller number is an upper bound of M. It is denoted as $\sup M$.

Similarly, we define the greatest lower bound (or infimum) of the set M (denoted as $\inf M$).

Problem 8 (✎). Prove that a number c is the supremum of the set M if and only if two conditions are satisfied:

1. For all $x \in M$, $x \leq c$.
2. For any number $c_1 < c$, there exists $x \in M$ such that $x > c_1$.

Problem 9. Can a set have multiple least upper bounds (or greatest lower bounds)?

Problem 10. Find $\sup M$ and $\inf M$ for:

1. $M = \{a^2 + 2a \mid -5 < a \leq 5\}$.
2. $M = \{\pm\frac{n}{2n+1} \mid n \in \mathbb{N}\}$.

Problem 11 (✎). Let A and B be subsets of $\mathbb{R}$, and let $\sup A$ and $\sup B$ be known.

1. Find $\sup(A \cup B)$.
2. Find $\sup(A + B)$, where $A + B = \{a + b \mid a \in A, b \in B\}$.
3. Find $\inf(A \cdot B)$, where $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$, if A and B consist of negative numbers.

Theorem 1 (Completeness Axiom)**.** *Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.*

Problem 12. Prove that $a < \sup A$ if and only if there is $c \in A$ such that $a < c$.

Problem 13. Prove the Completeness Axiom from our definitions.
Hint: define the digits of the supremum one by one.

Remark 2. Here we give real numbers constructively, but some textbooks use an axiomatic approach. The other axioms of real numbers are, roughly speaking, the axioms guaranteeing the usual properties of 4 arithmetic operations and Archimedes' Axiom, which guarantees that for any $0 < a < b$, there exists a natural n such that $an > b$. These axioms uniquely characterize what real numbers are.

Problem 14. Let $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots$ be an infinite sequence of closed line intervals. Prove that there exists a real c which lies in all of them. Is this true for open intervals?

Problem 15 (*)**.** Prove that the axiom of Archimedes follows from the Completeness Axiom.

Problem 16 (*)**.** Prove that if $c = \sup\,\{x \mid x^2 < 2\}$, then $c^2 = 2$.