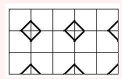
Game

Prepared by Nikita on June 16, 2023

Instructor's Handout

0-0

A checkered rectangle of size 10×12 was folded along the grid lines several times to form a 1×1 square. How many parts could there be if this square was cut along the segment connecting the midpoints of its two adjacent sides?



There could be 43, 37, 36, or 31 parts. Each cell will be cut exactly once, and the cut in one cell uniquely determines how the other cells are cut. We will get a cut of the type indicated in the diagram, and depending on the location of the cut-out small squares, we will get the number of parts $(67+1=43,\ 66+1=37,\ 57+1=36,\ 56+1=31)$.

0-1

What is the largest value for c for which the equation $x^2+6x+c=0$ has a solution?

$c=9$ because the discriminant must be non-negative, i.e., $D=6^2-4c \geq 0$.

0-2

Initially, all cells in a 3x3 table contain zeros. Several times, a 2x2 square is chosen and all the numbers within it are increased by 1. What number is written in the center of the table (see figure) if only the numbers in four cells of the original table are known?

	4	
5		
2		3

The number in the center of the table is 9. Each number in the middle of a side is the sum of the two numbers in the adjacent corner cells. Therefore, the number in the top-left corner is $5-2=3$, the number in the top-right corner is $4-3=1$. The number in the center is the sum of all the corner numbers, i.e., $3+1+2+3=9$.

0-3

The clover collection contains 30 rare clovers, they have 3, 4 or 5 leaves. The total amount of leaves is 100. Are there more 3-leaved clovers or 5-leaved clovers and by how much?

There are 20 less 5-leaved clovers than 3-leaved coins in the collection. Let x be the number of 3-leaved clovers, y be the number of 4-leaved clovers, and z be the number of 5-leaved clovers. From the given conditions, we obtain the system of equations: $x+y+z=30$ and $3x+4y+5z=100$. By subtracting four times the first equation from the second, we deduce that $z-x=-20$.

0-4
Find all the integer solutions of the equation

$$2^x \cdot (4 - x) = 2x + 4.$$

The solutions are $0, 1$, and 2. If $x > 4$ or $x < -2$, then the left and right sides of the equation have different signs. The remaining values can be checked.

0-5
What is a ones digit of a number $2023^2 + 2023^0 + 2023^2 + 2023^3$?

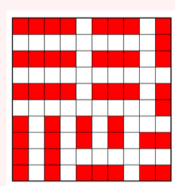
It is 6. Using modular arithmetic, we can figure out that it is the same as for the $3^2 + 1 + 3^2 + 3^3 = 46$.

0-6
Find the largest natural number with distinct digits such that any six consecutive digits form a number divisible by 6.

The largest number is 9753186420. This is because all digits starting from the sixth digit must be even, and this number is the largest among such numbers while satisfying the condition.

1-1
What is the maximum value of N for which it is possible to place N ships of sizes 1×4, 1×3, and 1×2 on a 10×10 grid? The ships cannot touch each other. Provide the answer and an example.

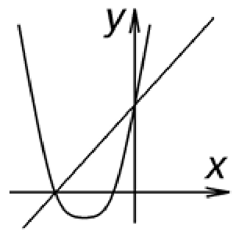
The maximum value of N is 5. Let's call the points of intersection of the grid lines *vertices*. A 1×4 ship occupies 10 vertices, a 1×3 ship occupies 8 vertices, and a 1×2 ship occupies 6 vertices (vertices cannot be shared between different ships as the ships do not touch each other). Therefore, the total number of vertices for all the ships is $N \cdot (10 + 8 + 6) = 24N$, which should not exceed the total number of vertices on the grid, which is $11 \times 11 = 121$. Hence, we have $24N \leq 121$, from which we get $N \leq 5$. An example for $N = 5$ is shown in the diagram.



1-2
The product of all natural divisors of a natural number n is equal to 2^{45}. Find n.

$2^9 = 512$. Since the product of all divisors of the desired number is a power of 2, the number itself is also a power of 2. Let's assume it is equal to 2^n. Then its divisors are $1, 2, 2^2, \ldots, 2^n$, and their product is equal to $2^{1+2+\ldots+n} = 2^{\frac{n(n+1)}{2}}$. Therefore, we have $\frac{n(n+1)}{2} = 45$, and solving this equation gives us $n = 9$.

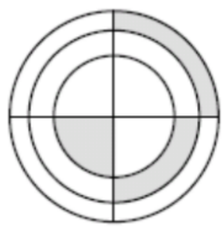
1-3
The graphs of the functions $y = 2x^2 + bx + c$ and $y = x + 1$ are shown on a diagram. Find b.



The solution is $b = 3$. By setting $x = 0$, we find the coordinates of the point of intersection with the y-axis to be $(0, 1)$, which means $c = 1$. From the equation of the line, we know that the coordinates of the point of intersection with the x-axis are $(-1, 0)$. Substituting this point into the equation of the parabola, we find that $b = 3$.

1-4

In the figure below, there are three concentric circles and two perpendicular diameters. If the three shaded figures have equal areas and the radius of the smallest circle is 1, what is the product of the three radii?



$\sqrt{6}$. Indeed, areas of the circles should have a ratio 1:2:3, so the radii are 1, 2 and $\sqrt{3}$.

1-5

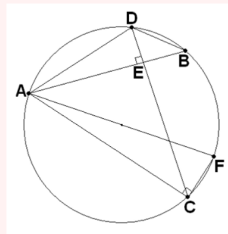
Point M lies inside an equilateral triangle and is located 1 cm, 2 cm, and 2 cm away from its sides. Find the area of the triangle.

The area of the triangle is $\frac{25\sqrt{3}}{3}$. The sum of these given lengths (5) is equal to the altitude of the triangle. Now it is easy to find the length of a side and the area.

1-6

(Archimedes' problem) "If chords AB and CD intersect at point E at a right angle, then the sum of the squares of the segments AE, BE, CE, and DE is equal to...." What is it equal to?

"... the square of the diameter of the circle." Let AF be the diameter. Since angle AED is a right angle, it is equal to angle ACF. But angle ADC = angle AFC, therefore in triangles ADE and AFC, angle DAE = angle CAF. Hence, arc CF is equal to arc BD, and therefore, CF = BD. Using the Pythagorean theorem, we have $AE^2 + CE^2 = AC^2, DE^2 + BE^2 = CF^2$, and $AC^2 + CF^2 = AF^2$. Thus, we obtain $AE^2 + CE^2 + DE^2 + BE^2 = AF^2$.



2-2

Find the largest natural number with distinct digits, such that the sum of its digits is divisible by the product of its digits.

The largest number is 321. This number has distinct digits, and the sum of its digits is 6, which is divisible by the product of its digits, which is 6 as well.

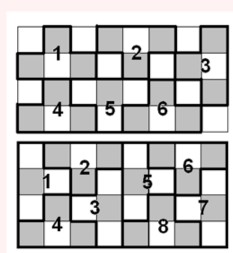
2-3

How many solutions does the puzzle have: $5 - U = C * I * R * C * L * E$? (The same letters represent the same digits, different letters represent different digits)

The puzzle has 8400 solutions. The digit U can only be 5, so in the right-hand side of the equation, there must be a 0 (5 options - any of the letters). The remaining 4 letters can take 4 values from the remaining 8 digits. Thus, there are 1680 (8!/4!) options for them. Therefore, there are a total of $5 \times 1680 = 8400$ solutions.

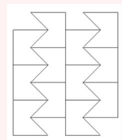
2-4

On a chessboard (without overlapping and following the grid lines), there are four-cell figures in the shape of the letter "T," covering all the black cells. How many of these figures can there be?



The possible numbers of figures are 12, 14, or 16. Each figure covers 1 or 3 black cells, which means an odd number of black cells. Since there is an even number of black cells on the board (32), the number of figures must be even. Additionally, the number of figures should be at least $\frac{32}{3}$ (each figure occupies at most 3 black cells) and at most $\frac{64}{4}$ (each figure occupies 4 cells of the entire board). Therefore, the total number of figures should be an even number greater than 10 and not exceed 16, i.e., 12, 14, or 16. All of these cases are possible because the black cells on half of the board in a 4×8 rectangle can be covered by either 6 or 8 figures (see the diagram).

2-5
Cover the plane with non-convex pentagons without overlaps.



For example, we can divide the plane into strips of equal width and then divide each strip into equal pentagonal ”flags”.

2-6
What is the largest value of N for which it is possible to place N black and N white kings on a chessboard such that black kings do not attack white kings and white kings do not attack black kings? Provide the answer and an example arrangement.

The largest value of N is 27. For example, the kings of one color can occupy the entire three bottom rows and the three leftmost squares of the fourth row, while the kings of the other color are arranged symmetrically to the kings of the first color with respect to the center of the board.

3-3
Find the largest natural number with distinct digits, such that the product of its digits is divisible by the sum of its digits.

The largest number is 9876543210. This number has distinct digits, and the product of its digits is 0, which is divisible by the sum of its digits, which is 45.

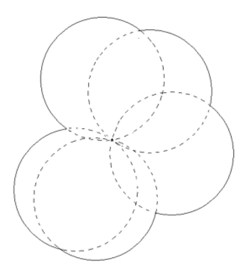
3-4
How many rational points lie on the sphere

$$(x-\sqrt{5})^2+(y-\sqrt{2})^2+(z-\sqrt{3})^2=10?$$

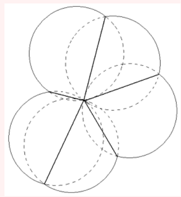
A rational point is a point where all three Cartesian coordinates are rational numbers.

There is 1 point with coordinates $(0,0,0)$. This can be proven by expanding the square, using natural transformations, and reasoning about rationality.

3-5
n paper circles
with a radius of 1 are arranged on a plane in such a way that their boundaries pass through a single point, and this point is located inside the entire region covered by the circles (see figure). This region forms a polygon with curvilinear sides. What can be the perimeter of such a curvilinear polygon?



The perimeter can be only 4π. The length of the arc of each circle that contributes to the perimeter is proportional to the angle it subtends. Since the sum of these inscribed angles is 2π, and the sum of the central angles that intercept these arcs is 4π, the total perimeter will be equal to twice the perimeter of one such circle, i.e., 4π.



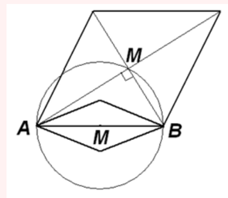
3-6
A simple magic square is defined as a 3×3 square grid in which there are 9 natural numbers (not necessarily distinct), and the sums of the numbers in each row and column are equal to each other. Find the largest value of n for which there exists a simple magic square containing the first n prime numbers. Provide the answer and an example of such a square.

The largest value of n is 6. Notice that there is exactly 1 even prime number, which is 2. Therefore, from the equality of the sums in each row and column, there must be at least 2 even numbers. Thus, we have $n \leq 7$. If $n = 7$, the table will contain the numbers 2, 3, 5, 7, 11, 13, 17, and exactly 2 even numbers. In this case, all 3 even numbers must be in different rows and columns, and each odd prime number (a) must be paired with another odd prime number (b) in two different ways to form the same sum with other numbers, i.e., $a + c = b + d$, $a + f = b + e$ (numbers a and b are in different rows and columns, and in the other intersections, even numbers are present). For the number 17, the possible pairs are $17 + 3 = 13 + 7$ and $17 + 7 = 13 + 11$, but in either case, the number 5 is not used, which is a contradiction. Hence, $n \leq 6$, and the square must contain the numbers 2, 3, 5, 7, 11, and 13. An example of such an arrangement can be seen in the diagram.

5	**7**	**11**
8	**13**	**2**
10	**3**	**10**

4-4
Points A and B are marked on a plane. Find the locus (all the possible positions) of the centers of rhombi with A and B as two of their vertices.

The locus is a circle with the diameter AB **and** its center, excluding points A and B. Let M be the center of a rhombus. Either M is the midpoint of AB, or angle AMB is 90 degrees. In the latter case, M belongs to the circle with AB as its diameter. However, M cannot coincide with points A and B.



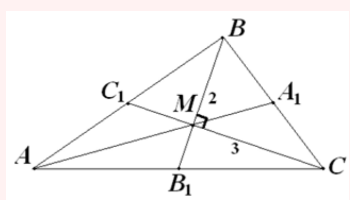
4-5

All natural numbers from 1 to n are written in a line in order. Under each pair of numbers in the next line, their sum is written. This process is repeated with the obtained line until only one number remains. Find this number.

The number is $(n+1)2^{n-2}$. Each number, k, in the first line contributes to the final number (sum S) with a coefficient equal to the number of paths from k to S. Let's consider the numbers k and n-k+1. Their sum is equal to n+1, and the number of paths from both of them to S is the same. Therefore, the desired number is n+1 divided by 2, times the number of paths from S to the first line, which is clearly equal to 2^{n-1}. Another solution approach is to denote the desired number as S_n. It is located in the n-th row. Notice that the numbers in the k-th row form an arithmetic progression with a common difference of 2^{k-1}. Hence, we have $S_{n+1} = 2S_n + 2^{n-1}$. The answer can be obtained easily using induction or other methods.

4-6

A median of a triangle is a segment connecting a vertex with a midpoint of an opposite side. Find the length of the third median of a triangle if the other two medians are perpendicular to each other and have lengths 2 and 3.



The length of the third median is $\sqrt{13}$. Let the medians $BB_1 = 2$ and $CC_1 = 3$, and let AA_1 be the third median, with M as the point of intersection of the medians. Considering the right triangle BCM, we can see that the segment MA_1 is equal to half the length of the hypotenuse BC, which can be found using the Pythagorean theorem. Taking into account that the point of intersection of the medians divides them in a ratio of $2:1$ from the vertex, we can find the length of the third median.

$$AA_1 = 3MA_1 = \frac{3}{2}BC = \frac{3}{2}\sqrt{BM^2 + CM^2} = \frac{3}{2}\frac{2}{3}\sqrt{2^2 + 3^2} = \sqrt{13}.$$

5-5

Find all pairs of quadratic equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ such that a and b are the roots of the second equation, and c and d are the roots of the first equation.

The pairs are (x^2+ax, x^2-ax), where a can be any number, and (x^2+x-2, x^2+x-2). Using Vieta's theorem, we have $a=-(c+d)$, $b=cd$, $c=-(a+b)$, and $d=ab$. From this system, we obtain $a+b+c=0$, $b=d$, and $b=bc=ab$. If $b=0$, then $d=0$ and $c=-a$, where a can be any number. If $b\neq 0$, then $a=c=1$, and $b=d=-2$.

5-6

The function $f(x)$ is defined for all real numbers x and satisfies the condition $2f(x)+f(1-x)=x^2$. Find all such $f(x)$.

$f(x)=\frac{x^2+2x-1}{3}$. Notice that $2f(x+1)+f(-x)=(x+1)^2$, and $2f(-x)+f(1+x)=x^2$. From this, we can deduce that $f(-x)=\frac{2x^2-(x+1)^2}{3}$.

6-6

Let $2S$ be the total weight of a set of weights. A number k is called nice if it is possible to choose k weights from the set such that their total weight is equal to S. What is the largest number of nice numbers that can be in a set of 10 weights? Provide the answer and an example of a set of 10 weights with the maximum number of nice numbers.

The maximum number of nice numbers is 7. An example of a set of 10 weights with the maximum number of nice numbers is: 1, 1, 2, 3, 5, 8, 13, 21, 34 (the first 9 Fibonacci numbers) and 20. We can form various combinations of weights to obtain the total weight $S=54$, such as: $34+20$, $13+21+20$, $5+8+21+20$, $2+3+8+21+20$, $1+1+3+8+21+20$, $1+1+2+3+5+8+34$, and $1+1+2+3+5+8+13+21$. The number 10 is not a nice number. If k is a nice number, then $(10-k)$ is also a nice number. If the number 1 is a nice number, then there is a weight of S in the set, and the only additional nice number will be 9. Therefore, there can be no more than 7 nice numbers (2, 3, 4, 5, 6, 7, and 8).