CIRCLES AND TABLES AND SLEEPING

JUNIOR CIRCLE 9/30/2012

1. Warm Up: The problem with sleeping Mathematicians

When we think about addition, we think about it occurring on a straight number line. However, we can think about addition on a circle instead. One easy example of this is on a clock. On an old clock, there are twelve numbers arranged in a circle.

Problem 1. Consider the clock with numbers $1, 2, \dots 12$ on it.

(a) Devon goes to bed at 8, and sleeps for 5 hours. What time will he wake up at? (Ignore AM/PM)

(b) Johnathan has a sleeping problem. He too goes to bed at 8, but he goes to sleep for 5 hours, and immediately goes back to sleep again for another 5 hours. He does this three times in total. What time will he wake up at?

(c) Jeff has an even larger sleeping problem. Jeff goes to sleep at 8 as well, but he goes to sleep for 5 hours and wakes up again a total of 85 times. What time will he wake up at?

(d) Morgan, while a perfectly good sleeper, has an obsession with his favorite number. Morgan also sleep for 5 hours a night, and goes to sleep at 8. When Morgan wakes up, he looks at his clock, and if the hour hand is pointing at his favorite number, he decides to get out of bed. Will Morgan ever be able to wake up?

(e) Isaac is just like Morgan, except he only sleeps for 4 hours at a time. He too goes to bed at 8, and will only get out of bed if he sees that the clock is showing his favorite number. Why should Jeff wake him up right now and tell him that this is a bad idea?

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2. Equivalence Classes in Modular Arithmetic

Let us recall what the definition of congruent $\mod n$. We say that two numbers, a and b are congruent $\mod n$ if their difference is divisible by n. For example, we have that

$$10 \equiv 24 \mod 7$$
$$4 \equiv 15 \mod 11$$

Congruent was an equivalence relation which meant that it satisfied three properties:

- **Reflexivity** $a \equiv a \mod n$
- **Symmetry** If $a \equiv b \mod n$ then $b \equiv a \mod n$
- **Transitivity** If $a \equiv b \mod n$ and $b \equiv c \mod n$ then $a \equiv c \mod n$

A **set** is a collection of objects, called **elements**. For example, all of the numbers form a set, and the students in this classroom form a set of people. If a set A is contained in another set B, then we say that A is a **subset** of B. Equivalence relations can break a set of objects being compared into smaller subsets. These subsets are given by things that are equivalent to eachother. We call each subset an **equivalence class**. This is a bit confusing, so it best to look at some examples.

(1) Let us say that two people in this classroom are equivalent if they are the same age. Then we have several subsets of equivalent people: those who are $1, 2, 3, 4, \ldots$ years old
(2) Let us say that two people are equivalent if they have the same colored hair. Then we have equivalence classes given by all of the people with blond hair, all of the people with black hair, all of the people with red hair, all of the people with brown hair, etc.
(3) Let us say that two numbers are equivalent if they are the same $\mod 5$. Then we have the equivalence classes of numbers that are congruent to $0 \mod 5$, $1 \mod 5$, $2 \mod 5$, $3 \mod 5$, and $4 \mod 5$.

Think of equivalence classes as bags, where we put equivalent things together. When we talk about equivalence classes, we need to put labels on the bags. How do we choose these labels? One way to label these bags is to choose some random item inside of it, and call the bag by the name of that item. For example, we might label the bag of people who are 21 years old in the classroom as **[Isaac]**. When we talk about the labels of bags, we put square brackets around the name of the label (so it kind of looks like a label).

Problem 2. Let our equivalence relationship be given congruence by $\mod 10$.

(a) Is 5 in the equivalence class $[14]_{10}$?

(b) Name all of the elements in the equivalence class $[2]_{10}$

(c) Can you find all the different equivalence classes for congruence $\mod 10$

Problem 3. Let $\sim$ be an equivalence relation.

(a) Show that if $a \sim b$ and $b \sim c$ and $c \sim d$, then $a \sim d$

(b) Show that if a is in the equivalence class $[b]$, and a is in an equivalence class $[c]$, then $[a] = [b]$

Problem 4. One good way to label equivalence classes $\mod n$ is with the smallest natural number in the bag. The natural numbers are $0, 1, 2, 3, \ldots$. For this problem, we work $\mod 5$

(a) Under this labeling method, what is a better label for $[22]_5$?

(b) How do you know that every bag can be labeled with a number less than 5? (Hint: Try using long division with remainders)

(c) Use your argument for the previous section to show that every number a is congruent $\mod 5$ to some number less than 5

3. Addition Multiplication Tables

Remember we also had that sums and multiplication worked with $\mod n$. If we had

$$a \equiv c \mod n$$

and

$$b \equiv d \mod n$$

then we have the congruences

$$a + b \equiv c + d \mod n$$

$$ab \equiv cd \mod n$$

This is just like regular multiplication and addition of whole numbers. However, there are a few differences: this kind of addition and multiplication is very interesting. For instance you can have two non-zero numbers multiply to 0. You can have even have a number which is it's own square!

How do we capture the structure of addition and multiplication? One way is to use addition and multiplication tables. We are all familiar with the usual addition and multiplication table:

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	8
3	3	4	5	6	7	8	9
4	4	5	6	7	8	9	10
5	5	6	7	8	9	10	11
6	6	7	8	9	10	11	12

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	8	10	12
3	0	3	6	9	12	15	18
4	0	4	8	12	16	20	24
5	0	5	10	15	20	25	30
6	0	6	12	18	24	30	36

However, one might consider looking at multiplication tables for modular arithmetic! This is a addition and multiplication table for adding and multiplying numbers $\mod 5$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Problem 5 (Even and Odds)**.** Using $\mod 2$, we can check if numbers are even or odd.

$\times_2$	0	1
0		
1		

$+_2$	0	1
0		
1		

(a) How do you use $\mod 2$ to check if a number is even or odd?

(b) Use an addition table for $\mod 2$ to show the sum of two even numbers is even. What about odd and even? Odd and odd?

(c) Use a multiplication table for $\mod 2$ to show that if $a \times b$ is odd, then a and b are both odd.

Problem 6 (Unusual square numbers)**.** Recall that we write x^2 for $x \times x$.

(a) Can you find all numbers x so that

$$x^2 \equiv x \mod 10$$

(b) What about all numbers so that

$$x^2 = x$$

(c) Can you find all numbers x other than 1 so that

$$x^2 \equiv 1 \mod 10$$

(d) What about all numbers x other than 1 so that

$$x^2 = 1$$

Problem 7 (Multiplying to 0)**.** (a) Are there any two non-zero numbers that multiply to 0? Give a proof of why or why not

(b) Can you find two numbers a and b so that

$$ab \equiv 0 \mod 10$$

(c) Find all solutions (where $0 \le a < 10$ and $0 \le b < 10$) to

$$ab \equiv 0 \mod 10$$

Problem 8. Construct a multiplication table and addition table for multiplying mod 3

$\times_3$	0	1	2
0			
1			
2			

$+_3$	0	1	2
0			
1			
2			

Problem 9. Construct a multiplication table and addition table for multiplying mod 4

$\times_3$	0	1	2	3
0				
1				
2				
3				

$+_3$	0	1	2	3
0				
1				
2				
3				

Problem 10. We say that a multiplication table is called **Sudoku** if no number occurs twice in any row or column, beside the 0 row or column.

(a) Can you find an example of a multiplication table $\mod n$ that is Sudoku for modular arithmetic? (Hint! Look through the tables that we have done)

(b) Can you find an example of a multiplication table that is not Sudoku?

(c) Suppose that b shows up twice in the same row a of a multiplication table. Show that b can be written as a product of a in two different ways.

(d) Can you show that if a table is not Sudoku, then there is a row with two zero's in it?

Problem 11. We say that an addition table is **Sudoku** if no number occurs twice in any row or column.

(a) Suppose that b shows up twice in the same row a of a addition table. Show that b can be written as a sum of a in two different ways.

(b) Can you find an addition table for modular arithmetic which is *not* Sudoku? Why or why not can you find one?

Problem 12. We have so far constructed multiplication tables for multiplication $\mod 2$, $\mod 3$, $\mod 4$ and $\mod 5$.

(a) Construct a multiplication table for $\mod 6$

$\times_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

(b) Which rows exhibit the numbers 0 through 5?

(c) Which rows do not?

Problem 13. Which numbers have multiplication tables that are Sudoku? Do you have a conjecture on which numbers have multiplication tables that are Sudoku?

Problem 14 (A return to the sleeping Mathematicians)**.** Jonathan, Isaac, Devon, Jeff and Morgan have gone to vacation in outer space!

(a) Devon plans on visiting the planet Mars. On Mars, they use clocks that have 7 hours on them. Devon decides on the following macabre sleep pattern: When he wakes up, he will look at his clock. He will then take whatever number he sees, and sleep for that many hours. If he wakes up at 7 oclock, he will get out of bed. Why is this a terrible idea?

(b) Isaac is on the planet Venus. On Venus, the clocks are only 6 hours long. Every time Isaac wakes up, he looks at his clock. If he has not seen that number before, he goes back to sleep. If Isaac wants to see every number on his clock, how long should each one of his naps be? (note: a proper nap is longer than 1 hour)

(c) Jeff is on the planetoid Pluto. Each day on Pluto is 6 hours long. Jeff starts walking around Pluto. As Pluto is not very big, it only takes Jeff 3 hours to walk around Pluto. Once every hour, Jeff looks up and sees if the sun is overhead. If Jeff starts with the sun directly overhead, how long will Jeff be walking before the sun is directly overhead again when he looks?

(d) Morgan is on the planet Jupiter. As it is quit a bit bigger than Pluto, it's day is 11 hours long. Morgan is like Devon, and is a bit rocket-lagged from his travels. When he wakes up, he looks at his clock, and sleeps for 3 times the amount that appears on his clock. Assuming Morgan starts at 2 o'clock, Does Morgan ever look at his clock and see 11 o'clock? What about 1 o'clock?

(e) Jonathan is back to his old tricks. After traveling to Saturn, Jonathan has decided to add a bit of randomness to his life. Saturn has a clock that is 17 hours long. When he goes to sleep, he constantly dreams of numbers. When he wakes up, he has a number x which is between 1 and 10 in his head. He looks at his clock, and sees the hour hand pointing to the number y. Jonathan then falls asleep for $x \times y$ hours. Jonathan plans on waking up at 17o'clock. Why should Morgan wake him up instead?