SPRING 2023

OLGA RADKO MATH CIRCLE

ADVANCED 1

JUNE 4TH, 2022

1. Quarter in Review

You MAY use your old packets. If you no longer have them, we don't have copies, but you can ask us to provide any definition or result that appears in the packet and we'll tell you.

Problem 1.1. Recall that our Turing machines use the symbols 1 and (blank), and upon reading symbol s_1 in state q_1, the machine does one of the following:

- Overwrite the symbol with s_2 and change the state to q_2 ($q_1 s_1 s_2 q_2$)
- Move one space left and change the state to q_2 ($q_1 s_1 L q_2$)
- Move one space right and change the state to q_3 ($q_1 s_1 R q_2$)
- Halt

To input n to a Turing machine, the machine starts with a contiguous string of $n+1$ 1's and begins reading the leftmost 1. The output of the Turing machine is the number of 1's when it halts (may not be contiguous).

(1) (3 points) Write a Turing machine that outputs 1 if the input is odd, and 0 if the input is even.
(2) (2 points) Write a Turing machine that halts if and only if the input is at least 2.

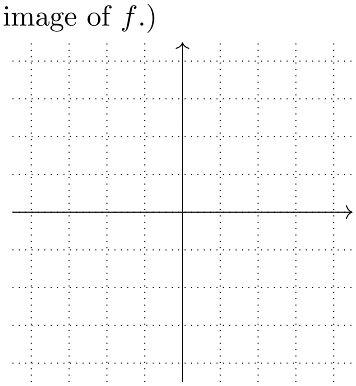
Problem 1.2. (2 points) Compute the following. (All must be correct for credit.)

(1) $3 \uparrow\uparrow\uparrow 1$
(2) $4 \uparrow\uparrow 2$
(3) $2 \uparrow\uparrow 4$

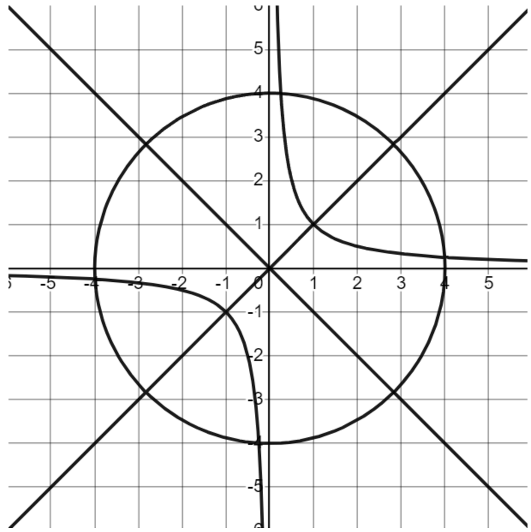
Problem 1.3. (2 points) Find the roots of the following polynomials, as well as their multiplicities. (All must be correct for credit.)

(1) $x^3 - 3x + 2$
(2) $x^6 + 3x^4 + 3x^2 + 1$

Problem 1.4. (2 points) Graph the image of the function $f(z) = 3z^2 + z$ in the complex plane, where the domain is the unit circle. (Your graph should pass through at least 8 marked points that are definitely in the image of f.)



Problem 1.5. (2 points) Write one equation $f(x, y) = 0$ whose set of solutions is the following basketball-like figure.



Problem 1.6. (2 points) Solve the following system of equations over projective space and compute the multiplicities of the intersections.

$$xy = 1$$
$$xy = -1$$

Problem 1.7. (2 points) Consider the number system $\mathbb{R}[x]$ defined as the set of all polynomials with real coefficients. (You might think it's strange to call a polynomial a "number", but you can add and multiply them all the same, and besides, isn't a real polynomial basically a number with digits in $\mathbb{R}$ and base x?) What are all of the units in this number system?

Problem 1.8. (2 points) Find the factorization of $6 + 7i$ into irreducibles over $\mathbb{Z}[i]$.

Problem 1.9. (2 points) Which of the following prime integers is a prime Gaussian integer?

$$7, 17, 37, 47, 67, 97, 107$$

Problem 1.10. (max 4 points) Secretly write down the largest number you can think of, defined from scratch. You can use what we covered in class (up arrows, busy beaver) for free, but everything else you must define. (That is, you cannot just write TREE(3) without first writing down what TREE is.) Your description is limited to one quarter of a sheet of 8.5" by 11" paper, one side.

Teams will be ranked at the end of this section by the size of their number. The first place team will get 4 points, second place 3 points, and so on (minimum 0 points).

2. Year in Review

Problem 2.1. (max 4 points) 30 minutes into this section, we will have a dotted line drawing competition. The goal is to draw a nice-looking dotted line on the chalkboard in one continuous motion. Each team nominates one person to draw a line, and you may practice as much as you want in your 30 minutes.

Teams will be ranked after all lines are drawn by popular vote. You may not vote for your own team. The first place team will get 4 points, second place 3 points, and so on (minimum 0 points).

Problem 2.2. ($\frac{1}{3}$ point for each correct placement, up to 3 points) Reorder the following sets from least to greatest cardinality. Write the smallest set to the left, largest to the right. If some sets have the same cardinality, write them above/below each other:

$$\{f : \mathbb{R} \to \mathbb{R}\},\ (1, 2),\ \mathbb{N},\ \mathbb{R},\ \mathbb{Z},\ [0, 3],\ \mathbb{Q},\ \{1, 2, 3\},\ \{f : \mathbb{N} \to \mathbb{N}\}$$

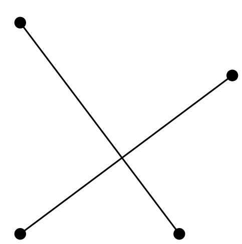
Problem 2.3. (2 points) How many proper expressions which use 5 pairs of parenthesis are there? For example, with 3 pairs, ()(()) is valid but ()())(is not.

Problem 2.4. (2 points) Oleg wrote ten letters to Math Circle parents and addressed the ten envelopes. However, he left the final stages of mailing to a careless secreatry who didn't pay attention, inserting the letters into the envelopes at random. (However, she did manage to fit exactly one letter in each envelope.) What is the probability that exactly nine of the ten letters is correctly addressed?

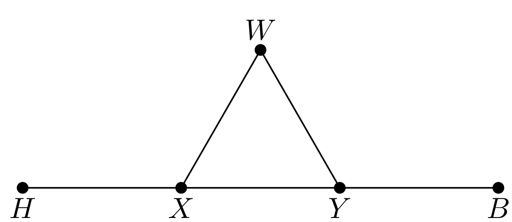
Problem 2.5. (2 points) Solve the following system of linear equations:

$$\begin{cases} x - 2y + 3z = 7 \\ 2x + y + z = 4 \\ -3x + 2y - 2z = -10 \end{cases} \tag{1}$$

Problem 2.6. (2 points) Describe the subset of the sphere that gives the following image under stereographic projection.



Problem 2.7. (2 points) A person starts from point Y on the following graph, and randomly walks along the graph, with an equal probability of travelling each edge. What is the probability that they reach home (vertex H) instead of falling into the all-consuming black hole (vertex B)?



Problem 2.8. A person on the Earth (modelled as a sphere) walks south 1 mile, then east 1 mile, then north 1 mile, and finds that they are exactly back where they started. Find *all* possible locations for this person's starting location.

Problem 2.9. (4 points) In fall quarter, we asked you to learn how to juggle 3 objects, and almost no one was successful. So let's take some baby steps.

Crumple a piece of paper into a ball. Successfully juggle 1 paper ball for 10 seconds. The key requirement: once the ball leaves one hand, the other hand may only move vertically to catch it, not left and right or forward and back. This is a key first step towards juggling 3 balls, because it is impossible to visually keep track of where every ball will land: you need to be confident in where the ball will land based just on how you threw it.

Problem 2.10. (4 points) You and your team will play a hat puzzle. Each person will be given a hat with a natural number on it (can be very large). You can see everyone else's hat but not your own. Then, order yourselves in a line (of your choosing). We will go down the line and ask each of you to guess the number on your hat. You may only guess natural numbers spoken with standard English readings. If at most one person guesses incorrectly, you win.

3. Compute π by experiment

The following items are available upon request from your instructors.

- String
- Uncooked grains of rice
- Compass
- Toothpicks
- Paper

In addition, unless otherwise stated in the problem, you may use:

- Ruler app on your phone
- Clock or stopwatch app on your phone
- Calculators and calculator apps on your phone
- Writing instruments such as a pens and pencils

We have limited quantities of physical rulers and stopwatches if you do not have a phone. Furthermore, in all experiments, here are the following important rules:

- You must have a written, approved experiment plan before you can start any experiment. The plan must be precise and you cannot deviate from your plan.
- Your plan must clearly list all the materials you are requesting, and you may not use any materials other than those listed.
- If your experiment plan is rejected, you may resubmit plans without penalty.
- All data collection must occur a fixed, finite number of times. For example, you may NOT write in your plan to "collect data until the average is between 3.141 and 3.142," or anything like that.
- You may not fake data. We're relying on your honor for this one, since it's probably not possible for your instructors to oversee all data collection.
- For any mathematical question (not an experiment), you can forfeit the question, get the answer from an instructor, and proceed with the experiment.
- Any experiment that works theoretically will be approved, but some will produce more reliable results than others. We will only approve one plan per question per team.

Problem 3.1. Compute π using its definition as the ratio of a circle's circumference to diameter.

(1) (2 points) Write a correct experiment plan and get it approved.
(2) (max 2 points) Teams will be ranked by the absolute value difference between their result and the true value of π. The first place team will get 2 points, second place 1.5 points, and so on (minimum 0 points).

Problem 3.2. Compute π using a process that (sufficiently) randomly picks a point in a particular area.

(1) (2 points) Write a correct experiment plan and get it approved.
(2) (max 2 points) Teams will be ranked by the absolute value difference between their result and the true value of π. The first place team will get 2 points, second place 1.5 points, and so on (minimum 0 points).

Problem 3.3. Compute π using Buffon's needle method. In this method, consider an infinite plane with lines spaced 1 unit apart. Let x be the average number of times a needle of length 1 dropped on the plane intersects one of these lines.

(1) (4 points) Compute x. (Hint: Would x change if the needle were circular? This is a challenging question, so remember you can forfeit it and do the experiment.)
(2) (2 points) Write a correct experiment plan and get it approved.

(3) (max 2 points) Teams will be ranked by the absolute value difference between their result and the true value of π. The first place team will get 2 points, second place 1.5 points, and so on (minimum 0 points).

Problem 3.4. Compute π using gravity. A pendulum is a weight attached to a string that rocks back and forth. The period of a pendulum is the time it takes to undergo one full back and forth cycle. Using differential equations and Newton's laws of motion, one can calculate that if the pendulum makes just a small angle,

$$T \approx 2\pi\sqrt{\frac{\ell}{g}},$$

where T is the period in seconds, ℓ is the length of the string in meters, and g is a constant that describes the acceleration on the surface of Earth due to gravity.

A very long time ago, people did not have standard units of measurement. A foot was the length of an actual person's foot and could vary greatly. But units of time were already widely standardized into hours, minutes, and seconds. In the 1600's, John Wilkins gave the first standardized definition of a meter: he defined it as the length of string that would make a pendulum swing from one end to the other in 1 second. (In other words, the period is 2 seconds.) Note that the modern definition of a meter is a slightly different length.

(1) (2 points) Based on the above information, find the relationship between π and g, using units as defined by Wilkins.

(2) (4 points) An object that begins to fall freely from rest at time $t = 0$ has height $y(t) = -\frac{1}{2}gt^2 + y_0$, where y_0 is the initial height. Using this, write a correct experiment plan to measure Wilkins' g and get it approved.

For this problem, you may NOT use a ruler, ruler app, or any pre-existing length-measuring device.

(3) (max 4 points) Teams will be ranked by the absolute value difference between their result and the true value of π. The first place team will get 4 points, second place 3 points, and so on (minimum 0 points).

Problem 3.5. Compute π using any algorithmic method. That is, you should write a formula or describe an algorithm that converges to π as you compute more and more. You may not look up any formulas, and you must be able to explain any formula you use. You may NOT use a calculator or any digital computational aids for this problem.

(1) (3 points) Write a correct computation plan and get it approved.

(2) (max 4 points) Teams will be ranked by the absolute value difference between their result and the true value of π. The first place team will get 4 points, second place 3 points, and so on (minimum 0 points).

Problem 3.6. (max 2 points) Recite π by whispering to an instructor for as many digits as you can. No studying or lookups allowed. You will be stopped after your first mistake. Teams will be ranked by number of digits recited. The first place team will get 2 points, second place 1.5 points, and so on (minimum 0 points). Ties will result in both teams receiving the average of the two places they occupy.