ORMC AMC 10/12B Final Contest

June 4, 2023

1. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90^\circ$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that the points M, N, A and C lie on a circle.

   Proof. Quadrilateral $ABCD$ is cyclic with diameter BD and center M, so $\angle AMC = 2\angle ADC$. Also N is the midpoint of QR and PS. Using right triangles PAS and QCR, we have

   $$\angle ANC = \angle ANP + \angle QNC = 2\angle ASP + 2\angle QRC = 2(\angle ASP + \angle DRS) = 2\angle ADC = \angle AMC.$$

   This concludes the proof. □

2. Find the smallest positive integer k for which there exists a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

   (a) For all positive integers m, n of the same colour, $f(m + n) = f(m) + f(n)$.

   (b) There are positive integers m, n such that $f(m + n) \neq f(m) + f(n)$.

   In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (a) and (b) the positive integers m, n are not necessarily distinct.

   Proof. The answer is $k = 3$. For $k = 3$, color each $n \in \mathbb{Z}_{>0}$ with the $n \pmod 3$ color and define

   $$f(n) = \begin{cases} n & 3 \mid n \\ 2n & 3 \nmid n. \end{cases}$$

   It is straightforward to verify that this coloring and this f does the job.

   We then prove that there is no coloring and f for $k = 2$. Suppose to the contrary. Observe first that $f(2n) = 2f(n)$ for any n.

   Let a be the smallest positive integer such that $f(a) \neq af(1)$. Note that a exists by condition (b). Also $a > 2$. Then a is odd, since otherwise $a/2$ is an integer less than a, so $f(a/2) = af(1)/2$. Then $f(a) = 2f(a/2) = af(1)$, a contradiction.

   Let $b < a$ be odd. If a and b have the same color, then $f(a + b) = f(a) + bf(1)$. On the other hand $a + b$ is even, and $(a + b)/2$ is an integer less than a, so $f(a + b) = 2f((a + b)/2) = (a + b)f(1)$. Then $f(a) = af(1)$, a contradiction. Thus any odd integer $b < a$ has a different color than a.

   Let $b < a$ be even. If a and b have different colors, then $a - b$ is odd and thus have different colors than a, so $a - b$ has the same color as b. Then $f(a) = f(a - b) + f(b) = af(1)$, a contradiction. Thus any even integer $b < a$ has the same color as a.

   In particular, $a - 1$ and a have the same color, and $a - 2$ and a have different colors. Also since $a + 1$ is even, we have $(a + 1)/2$ is an integer less than a, so $f(a + 1) = 2f((a + 1)/2) = (a + 1)f(1)$.

   Now suppose $a + 1$ has the same color as a. Then

   $$2f(a) = f(2a) = f(a + 1) + f(a - 1) = (a + 1)f(1) + (a - 1)f(1) = 2af(a),$$

which is a contradiction. Thus $a+1$ and a have different colors.

Since $a-2$ is an odd integer less than a, so $a-2$ and a have different colors, and then $a-2$ and $a+1$ have the same color. Then

$$f(a)+(a-1)f(1)=f(a)+f(a-1)=f(2a-1)=f(a-2)+f(a+1)=(a+1)f(1)+(a-2)f(1).$$

This implies $f(a)=af(1)$, a contradiction. Thus no such coloring and f exists for $k=2$. □

3. There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

   Proof. We first claim that we can color the regions bounded by the lines black and white, so that any two regions sharing a common side have different colors. Indeed, if there are no lines then the claim is trivial. Now suppose we have a coloring for $n-1$ lines. After adding the n-th line, we can keep the coloring on one side of the n-th line, and reverse the coloring on the other side of the n-th line. It is straightforward to verify that this coloring satisfies the conditions. Thus the claim follows by induction.

   Now suppose Turbo starts on a segment such that her left is a white region and her right is a black region (otherwise just reverse all colorings). Then each time no matter which direction she turns to, she still has on her left a white region and on her right a black region. Thus it is impossible for her to pass a line segment in both directions, since that means that she would have a white region on her right and a black region on her left. □

4. Let $n\geq 1$ be an integer and let $t_1<t_2<\cdots<t_n$ be positive integers. In a group of t_n+1 people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

   (i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.

   (ii) For every i with $1\leq i\leq n$, there is someone who has played exactly t_i games of chess.

   Proof. We proceed by induction. The case $n=1$ is clear since the complete graph on t_1+1 vertices works. Now suppose the claim holds for $n-1$. We first construct a graph G_1 with t_n-t_1+1 vertices and satisfies the conditions for $t_n-t_{n-1}<t_n-t_{n-2}<\cdots<t_n-t_1$. Then add in t_1 more isolated vertices, and take the complement graph. It is straightforward to verify that the final graph does the job. □

5. Let $n\geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is *expensive* if there exists a positive integer k such that

   $$(a_1+a_2)(a_2+a_3)\ldots(a_{n-1}+a_n)(a_n+a_1)=2^{2k-1}.$$

   a) Find all integers $n\geq 2$ for which there exists an expensive n-tuple.

   b) Prove that for every odd positive integer m there exists an integer $n\geq 2$ such that m belongs to an expensive n-tuple.

   There are exactly n factors in the product on the left hand side.

   (a) *Proof.* Tuple $(1, 1, \cdots, 1)$ does the job when n is odd. We then prove that there is no expensive tuple for even n. Proceed by induction.

When $n = 2$ the claim is clear since $(a_1 + a_2)(a_2 + a_1)$ is a perfect square but 2^{2k-1} is not. Now suppose to the contrary that there is an n-tuple $(a_1, \cdots, a_n)$. for some $n \geq 4$. Let $a_t = \max_{1 \leq i \leq n} a_i$. Then

$$a_{t-1} + a_t \leq 2a_t < 2(a_{t+1} + a_t)$$

and

$$a_{t+1} + a_t \leq 2a_t < 2(a_{t-1} + a_t)$$

where indices are considered modulo n. Note that $a_{t+1} + a_t$ and $a_{t-1} + a_t$ are powers of 2, so they are necessarily equal and then $a_{t-1} = a_{t+1}$. Then we can remove $a_{t+1} + a_t$ and $a_{t-1} + a_t$ from the product, and we are left with an expensive $(n-2)$-tuple, which contradicts our inductive hypothesis. Thus no expensive even tuples exist. □

(b) *Proof.* We prove by induction. We saw the case $m = 1$ in (a). Now suppose the claim holds for all $m' < m$. Write $2^k < m < 2^{k+1}$ for some integer k, and set $m' = 2^{k+1} - m < m$. Then we know that m' is in an expensive n-tuple $(a_1, \cdots, a_n)$ for some n. Assume $a_1 = m'$ after cyclicly permuting the indices. Then we see that $(m', m, m', a_2, \cdots, a_n)$ is an expensive $(n + 2)$-tuple. This completes the proof. □

6. Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2+1, n^2+2, \ldots, n^2+2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m+1, 7m+2, 7m+5, 7m+6$ and no elements of S of the form $7m + 3$ or $7m + 4$, where m is an integer.

Proof. The condition $n \in S$ is equivalent to $n^2 + k \mid n^4$ for some $1 \leq k \leq 2n$, which means

$$n^2 + k \mid n^4 - (n^2 + k)(n^2 - k) = k^2.$$

Let $k^2 = c(n^2 + k)$. Since $1 \leq k \leq 2n$, we have

$$4n^2 \geq k^2 = c(n^2 + k) \geq c(n^2 + 1).$$

This means c is 1 or 2 or 3.

When $c = 1$, we have $k^2 = n^2 + k$, so $(2n)^2 + 1 = (2k + 1)^2$. This is impossible for $n > 0$.

When $c = 2$, we have $k^2 = 2n^2 + 2k$, so $(k - 1)^2 - 2n^2 = 1$. This is a Pell's equation $x^2 - 2y^2 = 1$. The fundamental solution is $(x_1, y_1) = (3, 2)$, and we have the recurrence relation

$$\begin{aligned} x_{n+1} &= 3x_n + 4y_n \\ y_{n+1} &= 2x_n + 3y_n. \end{aligned}$$

Using $(x_1, y_1) = (3, 2)$, we can conclude all possibilities of $(x, y) \pmod 7$ are $(3, 2), (3, -2)$ and $(1, 0)$. Note that $y = n$. Thus n is of the forms $7m, 7m + 2$ and $7m + 5$ when $c = 2$, and there are infinitely many of each.

When $c = 3$, we have $k^2 = 3n^2 + 3k$. Thus $3 \mid k$, and then $9 \mid k^2 - 3k$, so $3 \mid n$. Write $k = 3k_0$ and $n = 3n_0$. Then we have $k_0^2 = 3n_0^2 + k_0$, or $(2k_0 - 1)^2 - 3(2n_0)^2 = 1$. This is a Pell's equation $x^2 - 3y^2 = 1$. The fundamental solution is $(x_1, y_1) = (2, 1)$, and we have the recurrence relation

$$\begin{aligned} x_{n+1} &= 2x_n + 3y_n \\ y_{n+1} &= x_n + 2y_n. \end{aligned}$$

Since x is odd and y is even, only (x_{2n}, y_{2n}) works. Using $(x_1, y_1) = (2, 1)$, we can conclude all possibilities of $(x_{2n}, y_{2n}) \pmod 7$ are $(0, 4)$, $(-1, 0)$, $(0, 3)$, and $(1, 0)$. Note that $y = 2n_0$ and $n = 3n_0$. Thus n is of the forms $7m + 6, 7m$ and $7m + 1$ when $c = 3$, and there are infinitely many of each. □