ORMC AMC 10/12B Final Contest

June 4, 2023

1. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90^\circ$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that the points M, N, A and C lie on a circle.

2. Find the smallest positive integer k for which there exists a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

   (a) For all positive integers m, n of the same colour, $f(m+n) = f(m) + f(n)$.

   (b) There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

   In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (a) and (b) the positive integers m, n are not necessarily distinct.

3. There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

4. Let $n \geq 1$ be an integer and let $t_1 < t_2 < \cdots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

   (i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.

   (ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

5. Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is *expensive* if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \ldots (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}.$$

   a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.

   b) Prove that for every odd positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

   There are exactly n factors in the product on the left hand side.

6. Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2+1, n^2+2, \ldots, n^2+2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m+1, 7m+2, 7m+5, 7m+6$ and no elements of S of the form $7m+3$ or $7m+4$, where m is an integer.