SPRING 2023 COMPETITION

OLGA RADKO MATH CIRCLE
INTERMEDIATE 2

1. Problems from Topics We Covered

1.1. Binary, Nim, and Nim-Sums.

Problem 1.1 (1 point). Find $10110 \oplus 11111$.

Solution. 01001.

Problem 1.2 (2 points). Find the winning move in the Nim game $(1, 2, 3, 6)$.

Solution. $6 \to 0$ as $1 \oplus 2 \oplus 3 = 0$.

1.2. Magic Squares.

Problem 1.3 (1 point). Two players take turns choosing one number at a time (without replacement) from the set $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$. The first player to obtain three numbers (out of three, four, or five) which sum to 0 wins. Does either player have a forced win?

Solution. No, this is the $1 - 9$ game.

1.3. Josephus Problem.

Problem 1.4 (1 point). Who survives in a Josephus problem scenario with $1,024$ people?

Solution. $\boxed{1}$ as 1024 is a power of 2.

Problem 1.5 (1 point). Who survives in a Josephus problem scenario with $1,027$ people?

Solution. $\boxed{7}$ as 1024 is a power of 2 and the difference is 3 so the survivor is $2(3)+1 = 7$.

1.4. Probability.

Problem 1.6 (1 point). A bag contains red, green, and blue marbles. If there are 17 red marbles and 9 blue marbles, and the probability of picking a green marble from the bag is $\frac{1}{3}$, then find the probability of picking a red marble.

Solution. There are 39 marbles in total, hence the probability of picking a red marble is $\frac{17}{39}$.

Problem 1.7 (3 points)**.** A soccer team of 5 players has won a trophy for a tournament and would like to take a picture. For this picture, all players will stand in order in a line, and one player will be designated to hold the trophy. How many such arrangements are possible?

Solution. $5 \cdot 5! = 600$

Problem 1.8 (2 points)**.** Find the coefficient of x^3y^8 in the polynomial $(x+y)^{11}$.

Solution. $\binom{11}{3} = \frac{11!}{3!8!} = 165$.

1.5. **15 Puzzle.**

Problem 1.9 (3 points)**.** In the following 15 Puzzle setup, is the game solvable?

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
15	**14**	**13**	

Solution. $\boxed{\text{No}}$: our faithful $\mathbb{Z}/2\mathbb{Z}$ invariant is equal to the sign of the permutation when the blank square is in the bottom right corner, and the given permutation is odd as it is a product of 3 transpositions.

1.6. **Game Theory.**

Problem 1.10 (3 points)**.** Find all Nash equilibria in the following game:

-6	6
-6	-12

Solution. This is equivalent to the Game 1 of Problem 6 of the first Game Theory packet, which has Nash Equilibrium -6 (top left corner) – only accept the answer if they specify top left, ask them which square if they do not specify any.

1.7. **Functional Equations.**

Problem 1.11 (2 points)**.** Find $f(10)$ if $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, and $f(1) = 1$.

Solution. 10. $f(1+1+\cdots+1) = f(1)+\cdots+f(1)$ so f is the identity on integers.

1.8. **Logarithms.**

Problem 1.12 (1 point)**.** Find $\log_{\sqrt[3]{3}}(27)$.

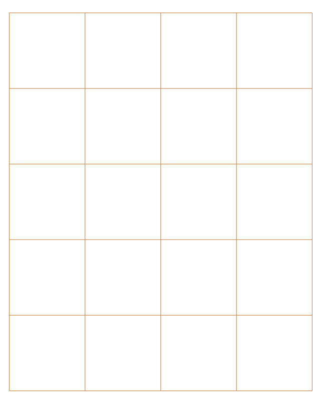
Solution. 9.

Problem 1.13 (3 points)**.** Recall that $\log(x) = \log_{10}(x)$ when no base is specified. Find $\log\left(\frac{1}{3}\right) + \log\left(\frac{3}{5}\right) + \log\left(\frac{5}{7}\right) + \cdots + \log\left(\frac{101}{103}\right)$.

Solution. $-\log(103)$ – the sum telescopes.

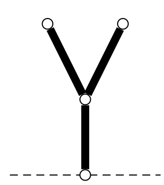
1.9. **Combinatorial Games.**

Problem 1.14 (2 points)**.** Assuming optimal play, who wins the following Chomp game?



Solution. Player 1 always wins from a rectangular position.

Problem 1.15 (3 points)**.** Evaluate the following Hackenbush tree.



Solution. 1; by the Colon Rule we can remove the two branches on top.

1.10. **Vieta's Formulas.**

Problem 1.16 (2 points)**.** Let $x_1, x_2, \ldots, x_{100}$ be the roots of $x^{100} - 37$. What is the product $x_1 \cdot x_2 \cdot \cdots \cdot x_{100}$?

Solution. -37 by Vieta's Formulas.

2. Miscellaneous Problems

Problem 2.1 (2 points)**.** Let ABC be a triangle such that point B lies inside the circle with diameter AC. What is the smallest possible value for $\angle ABC$?

Solution. The smallest possible value is when B lies exactly at the circle, and thus the smallest possible value is $90°$.

Problem 2.2 (3 points)**.** How many ways are there to write the numbers 0 through 9 in a row, such that each number other than the left-most is within one of some number to the left of it?

Solution. Observe that the sequence must terminate with a "0" or "9", and as we move from right to left, we always have a choice between writing the highest unused digit or the lowest—until we hit the left end, of course, where these two choices coincide. Thus, there are two choices at each of nine opportunities. It follows that the total number of ways is $2^9 = 512$.

Problem 2.3 (2 points)**.** What is $\frac{11!-10!}{9!}$?

Solution. 100.

Problem 2.4 (3 point)**.** For how many positive integers n is $n^2 - 3n + 2$ a prime number?

Solution.(AMC12, 2002) If $n \geq 4$, then $n^2 - 3n + 2 = (n-1)(n-2)$ is the product of two integers greater than 1, and thus is not prime. For $n \leq 3$, direct computation shows that $n^2 - 3n + 2$ is prime only when $n = 3$.

Problem 2.5 (4 points)**.** Find all integers n such that $\frac{n^2+3}{n-1}$ is an integer.

Solution. $\frac{n^2+3}{n-1} = \frac{(n^2-1)+4}{n-1} = n + 1 + \frac{4}{n-1}$. Thus $n - 1 = \pm 1, \pm 2, \pm 4$, so $n = 0, 2, -1, 3, -3, 5$.

Problem 2.6 (4 points)**.** There are three vertical pegs, and on one of them, there are 5 disks of increasing size (the smallest is on top). The goal is to move them all onto either of the other pegs. However, you may only move them one at a time, and no disk may ever be on top of a disk smaller than itself. What is the minimum number of moves needed to move all the disks?

Solution. 31 moves. For n disks, the minimum is $2^n - 1$ moves. This is the Tower of Hanoi problem. See `https://math.stackexchange.com/a/2660/` for solution details.

3. Difficult Problems

Problem 3.1 (4 points)**.** Find the number of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.

Solution. 40.

Problem 3.2 (4 points)**.** How many real numbers a are there so that the roots of $x^2 - ax + 2a = 0$ are integers?

Solution. 4.

Problem 3.3 (5 points)**.** What is the smallest integer less than or equal to $\frac{3^{100}+2^{100}}{3^{96}-2^{96}}$?

Solution. 80.

Problem 3.4 (4 points)**.** Compute the following sum:

$$\binom{100}{0}^2 + \binom{100}{1}^2 + \binom{100}{2}^2 + \ldots + \binom{100}{100}^2.$$

Solution. More generally, it is known that

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

So the answer is $\binom{200}{100}$.

Problem 3.5 (4 points)**.** Let $a_0 = 1$, and define $a_{n+1} = \sqrt{a_n + 1}$ recursively. What is a_∞? (That is, what number do I get if I apply this process an infinite number of times?)

Solution. ϕ, as if the limit exists, $\ell = \sqrt{\ell + 1} \implies \ell = \frac{1+\sqrt{5}}{2}$. The limit exists because $\sqrt{x+1}$ is a contraction when $x > 1$ so by the Contraction Mapping Theorem the iteration converges to the unique fixed point of the function when it's restricted to $[1, \infty) \to [1, \infty)$, but proving the limit exists is not required to solve this question.