Intermediate 1: Spring Relay!

50 Point Questions:

1. In how many ways can 1001 be written as the sum of two prime numbers?

2. A 3-inch long cube is made of 27 small cubes with edges of length 1-inch. 19 of these 27 cubes are white and the rest are black. Given that the 8 black cubes are placed in the corners of the large cube, what fraction of the surface area of the large cube is white?

3. James, Jim, Jimmy, Jimothy, and Jimtholomew are confused about today's day of the week. They make the following statements.

James:	"All I know is that yesterday was not Friday."
Jim:	"Tomorrow is Tuesday."
Jimmy:	"No, today is Tuesday."
Jimothy:	"The day after tomorrow is Tuesday"
Jimtholomew:	"Today is not Sunday, not Monday, and also not Tuesday".

   Only one of them is telling the truth. Which day is it today?

4. What is the probability that the product of two six-sided dice rolls is a multiple of 5?

5. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$? (if $2^8 + 1$ and $2^{18} + 1$ are cubes, you should count them too)

6. Consider the two operations:
$$a \diamond b = a^2 - b^2 \qquad \text{and} \qquad a \star b = (a - b)^2$$
   What is the value of
$$(5 \diamond 3) \star 6?$$

7. The 12-sided figure shown has been drawn on a 1 unit by 1 unit graph paper. What is the area of the figure in units squared?
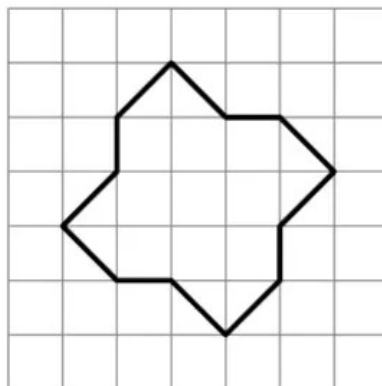
8. Sir Jimmothy has 10 unit blocks. He paints them red and then assembles them into the staircase shape below. Note that this staircase is laying on top of a table. Then, Sir Jimmothy decides to paint all visible sides of the staircase: the top, front, back, and both sides (but not the bottom!). After he is done, he disassembles the staircase. What total/cumulative surface area of the blocks will still be painted red?
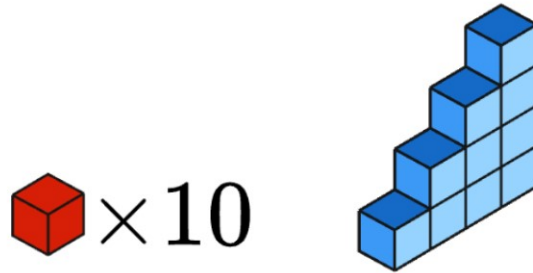
9. How many triangles below have an area equal to 1 and which ones are they?
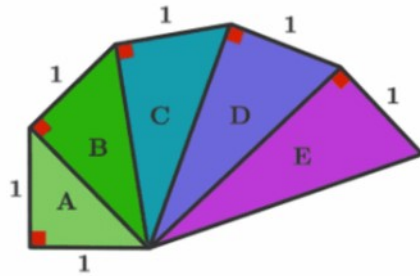

10. A palindrome is a number that reads the same forward and back. How many 3-digit palindromes are there?

100 Point Questions:

11. A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

12. The average value of 5 distinct, positive integers is 15, and the median is 18. What is the maximum possible value for the largest of these numbers?

13. Six boys live together in an apartment, which has two bathrooms. Each morning from 7:00 am they use both of the bathrooms before breakfast whereby they are 8, 10, 12, 17, 21, and 22 minutes respectively, constantly alone in one of the two bathrooms. What is the earliest time that all six boys can have breakfast together?

14. Jimothy's current age equals four times the age of his sister when she was twice as young as him. Fifteen years from now, the sum of the siblings' ages will be 100 years. How old is Jimothy now?

15. Sir Jimmothy the Honorable is thinking of a number n. Being an honorable sir, he has a fun riddle for you. All Jimothy will reveal is that if you triple n and then divide by 17, you will get a remainder of 13. He then asks you, what is the remainder you get when dividing n by 17?

16. Jimothy flips a fair coin 6 times. What is the probability that he gets more heads than tails?

17. What is the tens digit of 7^{2011}?

18. Triangle ABC is equilateral with the length of AB equal to 1. The lines DE and DG are both parallel to the line BC. Moreover, the triangle ADE, the trapezoid $DFGE$, and the trapezoid $FBCG$ all have the same perimeter. What is the length of DF plus the length of FG?
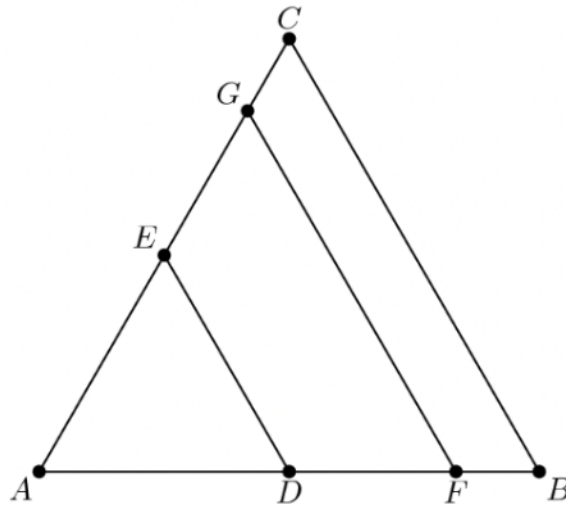
19. The diagram below shows a circle with center O as well as two straight lines: ABC and EDC. Suppose that $\angle AOE = 80°$ and $\angle BOD = 20°$. Find $\angle ACE$.
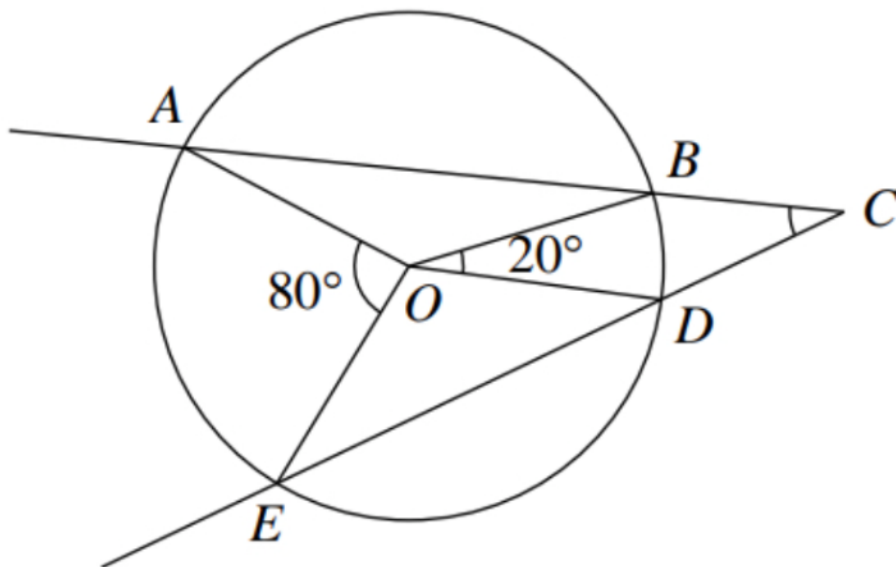
20. The integers from 1 to 99 are written down in ascending order without a gap:

$$123456789101112\cdots979899$$

This sequence of numbers is divided up into triples:

$$(123)(456)(789)(101)(112)\cdots(979)(899).$$

Circle whether or not each of the following triples is obtained in this process.

(a) (222)   YES / NO
(b) (444)   YES / NO
(c) (464)   YES / NO
(d) (646)   YES / NO
(e) (888)   YES / NO

200 Point Questions:

21. How many integers n between 1 and 2000 (including 1 and 2000) are such that $(2^n + 1) \mod 3 = 0$?

22. A grandmother, her daughter, and her granddaughter find that the sum of their ages is 100. Furthermore, each age is a power of two. How old is the granddaughter?

23. Let us define the following numbers

$$A = \underbrace{1111\cdots1111}_{\text{101 ones total}} \qquad \text{and} \qquad B = \underbrace{3333\cdots3333}_{\text{101 threes total}}.$$

Figure out the sum of the digits in the number $18 \times A \times B$.

24. Which positive integers n are such that 7 evenly divides $2^n + 6 \cdot 9^n$? To get you started, when $n = 1$ we have that

$$2^n + 6 \cdot 9^n = 2^1 + 6 \cdot 9^1 = 56$$

which is evenly divisible by 7 (meaning without remainder, since $56/7 = 8$). So $n = 1$ works!

25. 5 pirates of different ages have a treasure of 100 gold coins. On their ship, they decide to split the coins using this scheme: The oldest pirate proposes how to share the coins, and ALL pirates (including the oldest) vote for or against it. If 50% or more of the pirates vote for it, then the coins will be shared that way. Otherwise, the pirate proposing the scheme will be thrown overboard, and the process is repeated with the pirates that remain. As pirates tend to be a bloodthirsty bunch, if a pirate would get the same number of coins if he voted for or against a proposal, he will vote against so that the pirate who proposed the plan will be thrown overboard. Assuming that all 5 pirates are intelligent, rational, greedy, and do not wish to die, (and are rather good at math for pirates) what will happen?