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Some answers and hints

The following may be useful for solving **Problem 13**.

Let us find $3/4$ *mod* 7. To solve this problem, we need to think of division as of an operation opposite to multiplication. Multiplication undoes what division does (and the other way around). Three over four is such a number that it produces three when multiplied by four.

$$\frac{3}{4} \times 4 = 3$$

Similarly, in the *mod* 7 arithmetic,

$$\frac{3}{4} \times 4 \equiv 3 \; mod \; 7.$$

There are only seven congruence classes in the arithmetic, $0 \; mod \; 7$, $1 \; mod \; 7$, $2 \; mod \; 7$, $3 \; mod \; 7$, $4 \; mod \; 7$, $5 \; mod \; 7$, and $6 \; mod \; 7$. The class $3/4$ is one of them. Plugging the numbers $0, 1, 2, 3, 4, 5, 6$ into the expression

$$\square \times 4 \equiv 3 \; mod \; 7$$

one by one, we find that

$$6 \times 4 = 24 = 21 + 3 \equiv 3 \; mod \; 7.$$

Thus,

$$\frac{3}{4} \equiv 6 \; mod \; 7.$$

Solutions to **Problem 14**.

Find 3^{100} $(mod\ 7)$.

There are two ways to solve this problem. The first is more obvious, but requires more work. Let us take a look at the consecutive powers of three.

$$3^1 = 3 \equiv 3 \; mod \; 7$$

$$3^2 = 3^1 \times 3 = 9 \equiv 2 \; mod \; 7$$

$$3^3 = 3^2 \times 3 \equiv 2 \times 3 = 6 \equiv 6 \; mod \; 7$$

$$3^4 = 3^3 \times 3 \equiv 6 \times 3 = 18 \equiv 4 \; mod \; 7$$

$$3^5 = 3^4 \times 3 \equiv 4 \times 3 = 12 \equiv 5 \; mod \; 7$$

$$3^6 = 3^5 \times 3 \equiv 5 \times 3 = 15 \equiv 1 \; mod \; 7$$

$$3^7 = 3^6 \times 3 \equiv 1 \times 3 = 3 \equiv 3 \; mod \; 7 \equiv 3^1 \; mod \; 7$$

From this point on, the powers begin repeating one another in a cycle of length six.

$$3^8 = 3^7 \times 3 \equiv 3 \times 3 = 9 \equiv 2 \; mod \; 7 \equiv 3^2 \; mod \; 7$$

We get the following pattern.

$$3^1 \equiv 3^7 \equiv 3^{13} \equiv 3^{20} \ldots$$

$$3^2 \equiv 3^8 \equiv 3^{14} \equiv 3^{21} \ldots$$

$$3^3 \equiv 3^9 \equiv 3^{15} \equiv 3^{22} \dots$$
$$3^4 \equiv 3^{10} \equiv 3^{16} \equiv 3^{23} \dots$$
$$3^5 \equiv 3^{11} \equiv 3^{17} \equiv 3^{24} \dots$$
$$3^6 \equiv 3^{12} \equiv 3^{18} \equiv 3^{25} \dots$$

The final step of the solution is to figure out which of the above six sequences contains 3^{100}. Note that if we divide any power from the first sequence by six, the remainder will always be equal to 1. It will be equal to 2 for the second sequence, to 3 for the third, and so on. Now, $100 = 96 + 4 = 16 \times 6 + 4$. Therefore, 3^{100} will appear in the forth sequence.

$$3^{100} \equiv 3^4 \equiv 4 \ mod \ 7$$

The following way to solve the above problem is more elegant. Let us make a table with powers of two not exceeding 100 in the left column and with the number three raised to the corresponding power of two in the right one.

1	$3^1 = 3 \equiv 3 \ mod \ 7$
2	$3^2 = 3^1 \times 3^1 = 9 \equiv 2 \ mod \ 7$
4	$3^4 = 3^2 \times 3^2 \equiv 2 \times 2 = 4 \equiv 4 \ mod \ 7$
8	$3^8 = 3^4 \times 3^4 \equiv 4 \times 4 = 16 \equiv 2 \ mod \ 7$
16	$3^{16} = 3^8 \times 3^8 \equiv 2 \times 2 = 4 \equiv 4 \ mod \ 7$
32	$3^{32} = 3^{16} \times 3^{16} \equiv 4 \times 4 = 16 \equiv 2 \ mod \ 7$

64 $3^{64} = 3^{32} \times 3^{32} \equiv 2 \times 2 = 4 \equiv 4 \ mod\ 7$

128 $3^{128} = 3^{64} \times 3^{64} \equiv 4 \times 4 = 16 \equiv 2 \ mod\ 7$

Since $100 < 128$, we stop here. The last line is not needed for the subsequent computations. It is used as a stop sign.

Next, let us represent 100 as a sum of powers of two.

$$100 = 64 + 32 + 4$$

Therefore,

$$3^{100} = 3^{64} \times 3^{32} \times 3^{4} \equiv 4 \times 2 \times 4 = 32 \equiv 4 \ mod\ 7.$$

Problem 15 can be solved exactly the same way (any of the two) as Problem 14.

Problem 16. Find the last two digits of the following number.

$$7^{2012}$$

Since we are only interested in the last two digits, we can reformulate the problem as follows: find

$$7^{2012} \ mod\ 100.$$

Using either of the methods developed to solve Problem 14, we find

$$7^{2012} \equiv 1 \ mod\ 100.$$

Therefore, the last two digits of the number 7^{2012} are 01.