Combinatorics II

Math Circle (Intermediate)

October 7, 2012

If n is a natural number, then n! (pronounced "n factorial") is the product

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

For example, $2! = 2$; $3! = 6$, and $4! = 24$. We also define $0! = 1$.

1. Simplify the following expressions:

   (a)

   $$5! \cdot 6 \cdot 7 =$$

   (b) Express your answer using a factorial:

   $$2 \cdot 3 \cdot 4 =$$

   (c)

   $$\frac{27!}{25!} =$$

   (d) Write this product as a ratio of two factorials:

   $$n \cdot (n-1) = \frac{\quad !}{\quad !}$$

   (e) Simplify the fraction:

   $$\frac{n!}{(n-3)!} =$$

   (f) Simplify the fraction

   $$\frac{(2n-1)!}{(2n-3)!} =$$

(g) Simplify

$$\frac{17! \cdot 0!}{14! \cdot 3!} =$$

(h) Compare $\frac{(n+1)!}{(n-1)!}$ with n^2 for any positive integer n:

2. Find all possible values of k such that

$$\frac{(2k)!}{(2k-1)!} = \frac{(k+6)!}{(k+5)!}$$

3. (a) Last Sunday, at the end of class, all 29 students in the Intermediate group were to form a line at the front of the classroom. In how many ways could they have formed this line?

(b) At every meeting, Jeff selects 10 students (out of 29) from the class to form the line. He wants to have a different line every time. How many meetings in a row can they continuing forming new lines each time?

4. (a) How many "words" of length 3, consisting of all different letters, can you write in English?
   (Note that a "word" does not have to be in the dictionary. For example, we will count "abc" as a word.)

   (b) Express your answer to (a) as a ratio of two factorials.

$$\frac{\_\_\_\_!}{\_\_\_\_!}$$

5. (a) Sometimes, when I see a rainbow, certain colors are difficult to make out. If the colors of a full rainbow are red, orange, yellow, green, blue, indigo, and violet, but I only saw four colors, how many different color combinations could I have seen?

   (b) Express your answer to (a) as a ratio of two factorials.

$$\frac{\_\_\_\_!}{\_\_\_\_!}$$

6. (a) There are 10 students in a club. How many ways are there to pick a President and Vice President from the club's members?

   (b) Express your answer to (a) as a ratio of two factorials.

$$\frac{\_\_\_\_!}{\_\_\_\_!}$$

   (c) There are 10 students in another club. How many ways are there to select a committee of two people from the club's members?

   (d) Relate your answers to (a) and (c). Then explain (in a complete sentence, in writing!) why they differ. Use the words "distinct" and "identical" in your answer.

7. (a) How many ways are there to choose two objects out of n given ones, if the objects are distinct?

   (b) Express your answer to (a) as a ratio of factorials.

8. (a) How many ways are there to choose two objects out of n given ones, if the objects are identical?

   (b) Rewrite your answer to part (a) so it looks more similar to your result from Problem 7(b).

9. (a) How many ways are there to choose three objects out of n given ones, if the objects are distinct?

   (b) Express your answer to (a) as a ratio of factorials.

10. (a) How many ways are there to choose three objects out of n given ones, if the objects are identical?

(b) Rewrite your answer to part (a) so it looks similar to your result from Problem 7(b).

11. Generally, for problems concerning distinct objects, we use *permutations*. How many ways are there to choose k objects out of n given ones, *if the objects are distinct*?
Make sure to box your answer, because you have just derived an important formula!

12. Generally, for problems concerning identical objects, we use *combinations*. How many ways are there to choose k objects out of n given ones, *if the objects are identical*?
Make sure to box your answer, because you have just derived an important formula!

13. There are 20 students in a gym class. How many ways are there to pick a basketball team (consisting of 5 people)?

14. (a) A "necklace" is a circular string with several beads on it. It is allowed to rotate a necklace but not to turn it over. How many different necklaces can be made using 13 different beads?

    (b) Assume now that it is allowed to turn a "necklace" over. Now how many necklaces can be made using 13 different beads?

15. (Challenge problem) An artist is planning on mixing together any number of different colors from her palette. A mixture results as long as the artist combines at least two colors. If the number of possible mixtures is at most 200, what is the greatest number of colors the artist could have in her palette? (We assume that the colors that the artist has can not be obtained from each other by mixing).

ADDITIONAL CHALLENGE PROBLEMS:

16. There are six letters in the Hermetian language. A word is any sequence of six letters, some pair of which are the same. (The identical letters do not have to be next to each other in the word). How many words are there in the Hermetian language?

17. How many ways are there to place six rooks on a chessboard so that they do not attack each other?

18. Do 7-digit numbers having no digits 1 in their decimal representation constitute more than half of all 7-digit numbers?

19. You are making a bag consisting of 20 pieces of candy to give to your cousin for Halloween and can choose from 8 different types of candy. Assuming you give your cousin at least one of each types of candy, how many different bags could you make?

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[1]Some problems are taken from:
D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"