ORMC AMC 10/12 Training Week 7
Complex Numbers & Geometry

May 21, 2023

1 Warm-Up: Complex Number Review

1. **(2007 AMC 12 #18)** The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a + b + c + d$?

2. **(2009 AIME I #2)** There is a complex number z with imaginary part 164 and a positive integer n such that
$$\frac{z}{z+n} = 4i.$$
Find n.

3. **(1995 AIME #5)** For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

4. **(2005 AIME II #9)** For how many positive integers n less than or equal to 1000 is the following true for all real t:
$$(\sin t + i \cos t)^n = \sin nt + i \cos nt$$

5. Use the Cauchy-Schwarz inequality $|a||b| \geq |a \cdot b|$ to prove that for all complex numbers z, w:
$$|z + w| \leq |z| + |w|.$$
(Hint: start with $|z + w|^2$, and use the fact that $|z|^2 = z\overline{z}$)

2 Geometry and Complex Numbers

A lot of the properties of complex numbers make them a very natural tool to use in geometry, especially as an augmentation of coordinate geometry.

1. Any $z \in \mathbb{C}$ has *2 equivalent representations*: cartesian $(x+iy)$ and **polar** $(re^{i\theta} = r(\cos(\theta) + i\sin(\theta)))$. The cartesian representation is more similar to what we would use in standard coordinate geometry, but the polar form makes it much easier to "multiply" points, perform rotations, and work with angles in general, compared to standard coordinate geometry.

2. Recall that the complex conjugate $\overline{z}$ is the reflection of z across the x-axis. Together with rotations, this gives us a good way to think about and work with reflections in general.

3. Recall that we can represent lines by vectors, and vectors can easily be represented by complex numbers. For example, the vector $(a, b) \in \mathbb{R}^2$ corresponds to the complex number $z = a + bi$. If we let z, w represent vectors, they are parallel i.f.f z/w has no imaginary part, and they are perpendicular i.f.f. z/w has no real part. This can be generalized to find other angles, like $60°$ or $45°$.

4. The n^{th} roots of unity give us a natural way of thinking about regular polygons, since they are spaced evenly around the unit circle, creating a regular n-gon.

There are many more examples that could be listed. The point is that coordinate geometry gives us a methodical way to look at geometry problems, and the properties of complex numbers add various useful abilities to this toolkit.

2.1 Examples

1. The solutions of the equation $z^4 + 4z^3i - 6z^2 - 4zi - i = 0$ are the vertices of a convex polygon in the complex plane. What is the area of the polygon?

2. Notice that if $a = x_1 + iy_1$ and $b = x_2 + iy_2$, then $a\bar{b} - b\bar{a} = 2i(x_1y_2 - y_1x_2)$. Use this, and the shoelace theorem, to derive an area formula (similar to the shoelace theorem) for complex numbers.

   Recall that the shoelace theorem states that a polynomial with coordinate vertices (x_1, y_1), ... , (x_n, y_n) (listed in clockwise order) has area

   $$A = \frac{1}{2}\left|(x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)\right|$$

3. Show that complex numbers a, b, c, d lie on a circle if and only if $\dfrac{(c-a)/(c-b)}{(d-a)/(d-b)} \in \mathbb{R}$.

2.2 Exercises

1. Find the orthocenter of a triangle $\triangle ABC$ whose vertices can be represented by complex numbers a, b, c that lie on the unit circle. (We say that a, b, c are the ***affixes*** of A, B, and C, respectively).

2. The points $(0, 0), (a, 11), (b, 37)$ are vertices of an equilateral triangle. Find ab.

3. If $0, z_1, z_2$ form an equilateral triangle, show that $z_1^2 + z_2^2 = z_1 z_2$.

4. Use complex numbers to show that the midpoints of the sides of any quadrilateral form a parallelogram.

5. Prove the law of cosines using complex numbers.

6. Let A and B be arbitrary points in the plane. Show that the set of points M for which $\frac{MA}{MB} = k$, for some $k > 1$, is a circle.

7. Find the product of the lengths of all $\binom{n}{2}$ diagonals of a regular n-gon inscribed in a circle of radius 1.

8. Find the sum of the lengths of all $\binom{n}{2}$ diagonals of a regular n-gon inscribed in a circle of radius 1.

3 Bonus Exercises

1. Let $\zeta = e^{i\frac{2\pi}{5}}$. Show that $\zeta + \zeta^{-1} = \frac{\sqrt{5}-1}{2}$.
   (Hint: Recall that the sum of the n^{th} roots of unity is 0)

2. Find the exact values of $\cos(\frac{2\pi}{5})$ and $\sin(\frac{2\pi}{5})$.

3. Construct (or explain how to construct) a regular pentagon using only a compass and straightedge.

4. Prove Ptolemy's Inequality, which states that for any quadrilateral $ABCD$, we have

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD$$

5. Prove Ptolemy's Theorem, which states that equality holds in Ptolemy's Inequality (above) when the quadrilateral is cyclic.

6. A point A is taken inside a circle. For every chord of the circle passing through A, consider the intersection point of the two tangents at the endpoints of the chord. Find the set of these intersection points.