# Definable Sets

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## Part 1: Logical Algebra

#### **Definition 1:**

Logical operators operate on the values {True, False}, just like algebraic operators operate on numbers. In this handout, we'll use the following operators:

- $\neg$ : not
- $\wedge$ : and
- $\lor$ : or
- $\bullet \rightarrow: \mathrm{implies}$
- (), parenthesis.

The function of these is defined by *truth tables*:



 $A \wedge B$  is only true if both A and B are true.  $A \vee B$  is true when A or B (or both) are true.  $\neg A$  is the opposite of A, which is why it looks like a "negative" sign.

 $A \rightarrow B$  is a bit harder to understand. Read aloud, this is "A implies B."

The only time  $\rightarrow$  is false is when  $T \rightarrow F$ . This may seem counterintuitive, but it will make more sense as we progress through this handout.

#### Problem 2:

Evaluate the following.

- $\neg T$
- $\bullet \ F \vee T$
- $\bullet \ T \wedge T$
- $(T \wedge F) \vee T$
- $(T \wedge F) \vee T$
- $(\neg(F \lor \neg T)) \to T$
- $(F \to T) \to (\neg F \lor \neg T)$

#### Problem 3:

Evaluate the following.

- $A \to T$  for any A
- $(\neg(A \to B)) \to A$  for any A, B
- $(A \to B) \to (\neg B \to \neg A)$  for any A, B

**Problem 4:** Show that  $\neg(A \rightarrow \neg B)$  is equivalent to  $A \wedge B$ . That is, show that these give the same result for the same A and B. *Hint:* Use a truth table

**Problem 5:** Can you express  $A \lor B$  using only  $\neg$ ,  $\rightarrow$ , and ()?

Note that both  $\wedge$  and  $\vee$  can be defined using the other logical symbols. The only logical symbols we *need* are  $\neg$ ,  $\rightarrow$ , and (). We include  $\wedge$  and  $\vee$  to simplify our logical expressions.

## Part 2: Structures

#### **Definition 6:**

A *universe* is a set of meaningless objects. Here are a few examples:

- $\{a, b, ..., z\}$
- {0,1}
- $\mathbb{Z}$ ,  $\mathbb{R}$ , etc.

#### **Definition 7:**

A structure consists of a universe U and a set of symbols.

A structure's symbols give meaning to the objects in its universe.

Symbols come in three types:

- Constant symbols, which let us specify specific elements of our universe. Examples:  $0, 1, \frac{1}{2}, \pi$
- Function symbols, which let us navigate between elements of our universe. Examples:  $+, \times, \sin x, \sqrt{x}$
- Relation symbols, which let us compare elements of our universe. Examples:  $<, >, \leq, \geq$

The equality check = is **not** a relation symbol. It is included in every structure by default.

#### Example 8:

The first structure we'll look at is the following:

$$\Bigl(\mathbb{Z}\ \big|\ \{0,1,+,-,<\}\Bigr)$$

This is a structure with the universe  $\mathbb{Z}$  that contains the following symbols:

- Constants:  $\{0,1\}$
- Functions: {+, -}
  Relations: {<}</li>

If we look at our set of constant symbols, we see that the only integers we can directly refer to in this structure are 0 and 1. If we want any others, we must define them using the tools this structure offers. To "define" an element of a set, we need to write a sentence that is only true for that element. For example, if we want to define 2 in the structure above, we could use the sentence  $\varphi(x) = [1 + 1 = x]$ . Clearly, this is only true when x = 2.

## Problem 9:

Define -1 in  $(\mathbb{Z} | \{0, 1, +, -, <\})$ .

Let us formalize what we found in the previous two problems.

#### Definition 10:

A formula in a structure S is a well-formed string of constants, functions, and relations.

You already know what a "well-formed" string is: 1 + 1 is fine,  $\sqrt{+}$  is nonsense. For the sake of time, I will not provide a formal definition. It isn't particularly interesting.

A formula can contain one or more *free variables*. These are denoted  $\varphi(a, b, ...)$ . Formulas with free variables let us define "properties" that certain objects have. For example, x is a free variable in the formula  $\varphi(x) = [x > 0]$ .  $\varphi(3)$  is true and  $\varphi(-3)$  is false.

This "free variable" notation is much like the function notation you are used to:  $\varphi(x) = [x > 0]$  is similar to f(x) = x + 1, since the values of  $\varphi(x)$  and f(x) depend on x.

#### **Definition 11: Definable Elements**

Say S is a structure with a universe U. We say an element  $e \in U$  is *definable in* S if we can write a formula that only e satisfies.

#### Problem 12:

Define 2 in the structure  $(\mathbb{Z}^+ | \{4, \times\})$ . Hint:  $\mathbb{Z}^+ = \{1, 2, 3, ...\}$ . Also,  $2 \times 2 = 4$ .

**Problem 13:** Try to define 2 in the structure  $(\mathbb{Z} \mid \{4, \times\})$ . Why can't you do it?

**Problem 14:** What numbers are definable in the structure  $(\mathbb{R}^+_0 \mid \{1, 2, \div\})$ ?

## Part 3: Quantifiers

Recall the logical symbols we introduced earlier:  $(), \land, \lor, \neg, \rightarrow$ We will now add two more:  $\forall$  (for all) and  $\exists$  (exists).

#### Definition 15:

 $\forall$  and  $\exists$  are *quantifiers*. They allow us to make statements about arbitrary symbols.

Let's look at  $\forall$  first. Let  $\varphi(x)$  be a formula. Then, the formula  $\forall x \ \varphi(x)$  says " $\varphi$  is true for all possible x." For example, take the formula  $\forall x \ (0 < x)$ . In English, this means "For any x, x is bigger than zero," or simply "Any x is positive."

 $\exists$  is very similar: the formula  $\exists x \ \varphi(x)$  states that there is at least one x that makes  $\varphi$  true. For example,  $\exists (0 < x)$  means "there is a positive number in our set".

#### Problem 16:

Which of the following are true in  $\mathbb{Z}$ ? Which are true in  $\mathbb{R}_0^+$ ? *Hint:*  $\mathbb{R}_0^+$  is the set of positive real numbers and zero.

- $\forall x \ (x \ge 0)$
- $\neg(\exists x \ (x=0))$
- $\forall x \; [\exists y \; (y \times y = x)]$
- $[ \exists g \ (g \land g = x) ]$
- $\forall xy \exists z \ (x < z < y)$  This is a compact way to write  $\forall x \ (\forall y \ (\exists z \ (x < z < y)))$
- $\neg \exists x \ (\forall y \ (x < y))$

#### **Problem 17:** Does the order of $\forall$ and $\exists$ in a formula matter? What's the difference between $\exists x \ \forall y \ (x \leq y)$ and $\forall y \ \exists x \ (x \leq y)$ ? *Hint:* In $\mathbb{R}^+$ , the first is false and the second is true. $\mathbb{R}^+$ does not contain zero.

Problem 18: Define 0 in  $\left(\mathbb{Z} \mid \{\times\}\right)$ 

Problem 19: Define 1 in  $\left(\mathbb{Z} \mid \{\times\}\right)$ 

Problem 20: Define -1 in  $\left(\mathbb{Z} \mid \{0, <\}\right)$ 

 $\begin{array}{l} \textbf{Problem 21:} \\ \text{Let } \varphi(x) \text{ be a formula.} \\ \text{Define } (\forall x \ \varphi(x)) \text{ using logical symbols and } \exists. \end{array}$ 

## Part 4: Definable Sets

Armed with  $(), \land, \lor, \neg, \rightarrow, \forall$ , and  $\exists$ , we have enough tools to define sets.

#### **Definition 22: Set-Builder Notation**

Say we have a condition c. The set of all elements that satisfy that condition can be written as follows:

 $\{x \mid c \text{ is true}\}$ 

This is read "The set of x where c is true" or "The set of x that satisfy c."

For example, take the formula  $\varphi(x) = \exists y \ (y + y = x)$ . The set of all even integers can then be written

$$\{x \mid \varphi(x)\} = \{x \mid \exists y \ (y+y=x)\}$$

#### **Definition 23: Definable Sets**

Let S be a structure with a universe U. We say a subset M of U is *definable* if we can write a formula that is true for some x iff  $x \in M$ .

For example, consider the structure  $\left(\mathbb{Z} \mid \{+\}\right)$ 

Only even numbers satisfy the formula  $\varphi(x) = \exists y \ (y + y = x)$ , So we can define "the set of even numbers" as  $\{x \mid \exists y \ (y + y = x)\}$ . Remember—we can only use symbols that are available in our structure!

#### Problem 24:

Is the empty set definable in any structure?

Problem 25: Define  $\{0,1\}$  in  $\left(\mathbb{Z}_0^+ \mid \{<\}\right)$ 

Problem 26:

Define the set of prime numbers in  $\left(\mathbb{Z} \mid \{\times, \div, <\}\right)$ 

#### Problem 27:

Define the set of nonreal numbers in  $(\mathbb{C} \mid \{\operatorname{real}(z)\})$ Hint: real(z) gives the real part of a complex number: real(3 + 2i) = 3 Hint: z is nonreal if  $x \in \mathbb{C}$  and  $x \notin \mathbb{R}$ 

Problem 28: Define  $\mathbb{R}_0^+$  in  $\left(\mathbb{R} \mid \{\times\}\right)$ 

**Problem 29:** Let  $\triangle$  be a relational symbol.  $a \triangle b$  holds iff a divides b. Define the set of prime numbers in  $(\mathbb{Z}^+ | \{ \triangle \})$ 

#### Theorem 30: Lagrange's Four Square Theorem

Every natural number may be written as a sum of four integer squares.

Problem 31: Define  $\mathbb{Z}_0^+$  in  $\left(\mathbb{Z} \mid \{\times, +\}\right)$ 

Problem 32: Define < in  $\left(\mathbb{Z} \mid \{\times, +\}\right)$ 

*Hint:* We can't formally define a relation yet. Don't worry about that for now. You can repharase this question as "given  $a, b \in \mathbb{Z}$ , can you write a sentence that is true iff a < b?"

#### Problem 33:

Consider the structure  $S = (\mathbb{R} \mid \{0, \diamond\})$ The relation  $a \diamond b$  holds if |a - b| = 1

Part 1: Define  $\{-1, 1\}$  in S.

### Part 2:

Define  $\{-2, 2\}$  in S.

#### Problem 34:

Let P be the set of all subsets of  $\mathbb{Z}_0^+$ . This is called a *power set*. Let S be the structure  $(P \mid \{\subseteq\})$ 

#### Part 1:

Show that the empty set is definable in S. *Hint:* Defining {} with  $\{x \mid x \neq x\}$  is **not** what we need here.

We need  $\emptyset \in P$ , the "empty set" element in the power set of  $\mathbb{Z}_0^+$ .

#### Part 2:

Let  $x \approx y$  be a relation on P.  $x \approx y$  holds if  $x \cap y \neq \{\}$ . Show that  $\approx$  is definable in S.

**Part 3:** Let f be a function on P defined by  $f(x) = \mathbb{Z}_0^+ - x$ . This is called the *complement* of the set x.

## Part 5: Equivalence

#### Notation:

Let S be a structure and  $\varphi$  a formula. If  $\varphi$  is true in S, we write  $S \models \varphi$ . This is read "S satisfies  $\varphi$ "

### **Definition 35:**

Let S and T be structures. We say S and T are *equivalent* and write  $S \equiv T$  if for any formula  $\varphi$ ,  $S \models \varphi \iff T \models \varphi$ . If S and T are not equivalent, we write  $S \not\equiv T$ .

Problem 36: Show that  $\left(\mathbb{Z} \mid \{+,0\}\right) \not\equiv \left(\mathbb{R} \mid \{+,0\}\right)$ 

Problem 37: Show that  $\left(\mathbb{Z} \mid \{+,0\}\right) \not\equiv \left(\mathbb{N} \mid \{+,0\}\right)$ 

Problem 38: Show that  $(\mathbb{R} \mid \{+, 0\}) \not\equiv (\mathbb{N} \mid \{+, 0\})$ 

Problem 39: Show that  $(\mathbb{R} \mid \{+, 0\}) \not\equiv (\mathbb{Z}^2 \mid \{+, 0\})$ 

Problem 40: Show that  $\left(\mathbb{Z} \mid \{+,0\}\right) \not\equiv \left(\mathbb{Z}^2 \mid \{+,0\}\right)$