ORMC AMC 10/12 Training Week 7
Number Theory III

May 14, 2023

1 Pell's Equation

Recall that a **Pell's equation** is an equation of the form

$$x^2 - dy^2 = 1 \tag{1}$$

for any integer d that is not a perfect square.

Theorem 1. When d is not a perfect square, the Pell's equation $x^2 - dy^2 = 1$ has infinitely many solutions generated by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n \tag{2}$$

where (x_1, y_1) is the *fundamental solution*, i.e., the nontrivial solution (x, y) with smallest $x + y\sqrt{d}$.

Usually we also want integral properties of solutions of Pell's equations, such as how they behave when modulo a number. From (2), we can deduce an integral recurrence relationship of all solutions of (1).

$$\begin{aligned} x_{n+1} &= x_1x_n + dy_1y_n, \\ y_{n+1} &= y_1x_n + x_1y_n. \end{aligned}$$

An variant of Pell's equation is

$$x^2 - dy^2 = -1 \tag{3}$$

for any integer d that is not a perfect square. This type of equations is not guaranteed to have solutions, even if when d is not a perfect square. However, if there is a solution, then there are infinitely many solutions generated by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^{2n-1} \tag{4}$$

where (x_1, y_1) is the fundamental solution.

1. **(ARML)** Let n be a positive integer, and consider the list $1, 2, 2, 3, 3, 3, ..., n, n, ..., n$ where the integer k appears k times in the list for $1 \leq k \leq n$. The integer n will be called "ARMLy" if the median of the list is not an integer. The least ARMLy integer is 3. Compute the least ARMLy integer greater than 3.

2. Prove that if n is a natural number and $(3n + 1)$ and $(4n + 1)$ are both perfect squares, then 56 will divide n.

3. **(AIME)** Find the largest integer n satisfying the following conditions:
   (i) n^2 can be expressed as the difference of two consecutive cubes;
   (ii) $2n + 79$ is a perfect square.

4. **(British Math Olympiad)** Find the first integer $n > 1$ such that the average of $1^2, 2^2, \ldots, n^2$ is itself a perfect square.

5. **(European Girls Math Olympiad)** Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2 + 1, n^2 + 2, \ldots, n^2 + 2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of S of the form $7m + 3$ or $7m + 4$, where m is an integer.

2 Arithmetic Functions

An arithmetic function is simply any function $f : \mathbb{Z} \to \mathbb{C}$. We say an arithmetic function f is **multiplicative**, if for any integers a and b with $\gcd(a, b) = 1$, there is $f(ab) = f(a)f(b)$. We introduce 3 most important arithmetic functions here.

1. The function $\tau(n)$ is defined to be the number of positive divisors of an integer n.
2. The function $\sigma(n)$ is defined to be the sum of all positive divisors of an integer n.
3. The Euler Phi function $\varphi(n)$ is defined to be the number of positive integers less than n that are relatively prime to n.

Theorem 2. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n. Then

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$$

$$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

In particular, they are multiplicative functions.

3 Examples

1. Prove the expression for $\sigma(n)$ in the theorem, i.e.,

   $$\prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = \sum_{d|n} d$$

   where the summation means summing over all positive divisors of n.

2. Find all n such that $\varphi(n) = 12$.

3. Find the sum of even positive divisors of 10000.

3.1 Exercises

1. What is the sum of the positive integer divisors of 8128?

2. Find all positive integers n such that sum of all its positive divisors is $n^2 - 6n + 5$.

3. Show that for positive integers m and n, there is $\tau(mn) \leq \tau(m)\tau(n)$.

4. Determine the product of all distinct positive integer divisors of $n = 420^4$.

5. Using the same method as problem 3, conclude that for any positive integer n,

$$\prod_{d|n} d = n^{\frac{\tau(n)}{2}}.$$

6. Let $k > 1$ be such that $2^k - 1$ is a prime number. Show that
   (a) k is prime.
   (b) $n = 2^{k-1}(2^k - 1)$ is a *perfect number*, i.e., it is equal to the sum of its proper divisors.
   (c) The product of the positive divisors of n is n^k.

7. Let n be a positive integer such that the sum of its positive divisors is at least $2022n$. Prove that n has at least 2022 distinct prime factors.

4 Bonus Exercises

1. Determine the number of ordered pairs of positive integers (a, b) such that the least common multiple of a and b is $2^3 5^7 11^{13}$.

2. Using the same method as problem 1, conclude that if $n = p_1^{a_1} \cdots p_k^{a_k}$ is a prime decomposition of n, then there are $(2a_1 + 1)(2a_2 + 1)(2a_k + 1)$ distinct pairs of ordered positive integers (a, b) with $\text{lcm}(a, b) = n$.

3. Use AM-GM to show that for any positive integer n, $\tau(n) \leq 2\sqrt{n}$.

4. Let $n > 1$ be any integer. Show that
$$\sum_{d|n} \frac{d}{\sqrt{n}} = \sum_{d|n} \frac{\sqrt{n}}{d}.$$

5. Show that
$$\sum_{d|n} \varphi(d) = n.$$