Spring Quarter - Worksheet 6: Transformations of the projective spaces

In the previous lectures, we studied projective lines \( \mathbb{P}^1_\mathbb{R} \) and projective planes \( \mathbb{P}^2_\mathbb{R} \) in a projective space \( \mathbb{P}^3_\mathbb{R} \). In this worksheet, we will learn some applications of projective geometry. First, we will start by recalling concepts and solving preliminary problems.

The projective plane \( \mathbb{P}^2_\mathbb{R} \) has points with coordinates \([x : y : z]\) and lines given by equations \(ax + by + cz = 0\). The dual projective plane \( \mathbb{P}^2_\mathbb{R}^\vee \) will be the space parametrizing lines in \( \mathbb{P}^2_\mathbb{R} \). More precisely, a line \(ax + by + cz = 0\) in \( \mathbb{P}^2_\mathbb{R} \), will correspond to the point \([a : b : c]\) in \( \mathbb{P}^2_\mathbb{R}^\vee \).

**Problem 6.0** For the following lines in \( \mathbb{P}^2_\mathbb{R} \): Write down the coordinates of the corresponding points in \( \mathbb{P}^2_\mathbb{R}^\vee \).

Draw the lines in \( \mathbb{P}^2_\mathbb{R} \) and the corresponding points in \( \mathbb{P}^2_\mathbb{R}^\vee \).

In \( \mathbb{P}^2_\mathbb{R}^\vee \) identify which triples of points are colinear.

- \( l_0 : \{x = 0\}\)
- \( l_1 : \{y = 0\}\)
- \( l_2 : \{z = 0\}\)
- \( l_3 : \{x + y = 0\}\)
Problem 6.1 Show that three lines \( a_0x + b_0y + c_0z = 0, \ a_1x + b_1y + c_1z = 0 \) and \( a_2x + b_2y + c_2z = 0 \) in \( \mathbb{P}^2_\mathbb{R} \) intersect at a single point if and only if the corresponding points \( ([a_0 : b_0 : c_0], \ [a_1 : b_1 : c_1] \) and \( [a_2 : b_2 : c_2] \) of \( \mathbb{P}^2_\mathbb{R} \) are colinear.

What is the relation between the common point in \( \mathbb{P}^2_\mathbb{R} \) and the common line in \( \mathbb{P}^2_\mathbb{R} \)?
As we established before, we will use coordinates \([x : y : z]\) for \(\mathbb{P}^2_\mathbb{R}\) and coordinates \([a : b : c]\) in \(\mathbb{P}^2_\mathbb{R}^{2\vee}\). We can assign a correspondence between points \([x : y : z]\) in \(\mathbb{P}^2_\mathbb{R}\) and lines \(xa + yb + zc = 0\) in \(\mathbb{P}^2_\mathbb{R}^{2\vee}\). Notice that the line \(xa + yb + zc = 0\) has \(a, b\) and \(c\) as variables and \(x, y\) and \(z\) as constants.

**Problem 6.2** For the following points in \(\mathbb{P}^2_\mathbb{R}\), write down the equations of the corresponding lines in \(\mathbb{P}^2_\mathbb{R}^{2\vee}\).

- \(p_1 : [1 : 1 : 1]\)
- \(p_2 : [1 : 0 : 1]\)
- \(p_3 : [0 : 1 : 1]\)
- \(p_4 : [1 : 1 : 0]\)
For a curve $C$ with equation $f(x, y, z) = 0$ in $\mathbb{P}_k^2$, with a point $[x_0, y_0, z_0]$ inside the curve. The equation of the tangent line to $C$ at the point $p$ is:

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)x + \frac{\partial f}{\partial y}(x_0, y_0, z_0)y + \frac{\partial f}{\partial z}(x_0, y_0, z_0)z = 0$$

**Problem 6.3** For the following curves and points compute the tangent lines:

- $C := \{x^2 + y^2 - z^2 = 0\}, p := [0 : 1 : 1], q := [3 : 4 : 5]$
- $C := \{x = 0\},$ any point.
- $C := \{x^3 + y^3 + z^3 = 0\}, p := [0 : 1 : -1], q := [1 : 0 : -1]
Now we will define the dual of a general curve in $\mathbb{P}^2_k$.

For each point $p$ in a curve $C$ in $\mathbb{P}^2_k$, we can take the tangent line to $C$ at $p$. These tangent lines correspond to points in $\mathbb{P}^2_k$. These points in $\mathbb{P}^2_k$ form a curve $C^\vee$, called the dual curve of $C$.

If $C$ is a smooth curve of degree $d$, then the curve $C^\vee$ is a curve of degree $d(d - 1)$

**Problem 6.4** Using the fact that the following conics are smooth, compute the defining function of $C^\vee$:

- $C := \{x^2 + y^2 - z^2 = 0\}$
- $C := \{xz - y^2 = 0\}$
- $C := \{xy + yz + zx = 0\}$

Hint: You can use that five points define a conic.
Let $C$ be a curve given by an equation of the form $Px^2 + Qy^2 + Rz^2 = 0$. The equation of the tangent line to $[x_0 : y_0 : z_0]$ is $2Px_0x + 2Qy_0y + 2Rz_0z = 0$. Hence the points in $C^\vee$ are of the form $[2Px_0 : 2Qy_0 : 2Rz_0] = [Px_0 : Qy_0 : Rz_0]$.

Since $[x_0 : y_0 : z_0]$ is in $C$, we have that $Px_0^2 + Qy_0^2 + Rz_0^2 = 0$. Therefore the points in $C^\vee [a : b : c] = [Px_0 : Qy_0 : Rz_0]$ are of the form $a^2 = P^2x_0^2$, $b^2 = Q^2y_0^2$ and $c^2 = R^2z_0^2$.

Therefore the equation of $C^\vee$ is $\frac{1}{P}a^2 + \frac{1}{Q}b^2 + \frac{1}{R}c^2 = 0$. Remember that in $\mathbb{P}_R^2$, $a$, $b$ and $c$ are the variables.

**Problem 6.5** Following the method for curves of the form $Px^2 + Qy^2 + Rz^2 = 0$, compute the equation of the curve $C^\vee$ for the following conics:

- $Pxy + Qz^2 = 0$
- $Pxy + Qyz + Rzx = 0$

Verify that these general formulas were achieved in the particular cases of Problem 6.4
A line arrangement in $\mathbb{P}_R^2$ is simply a collection of lines in $\mathbb{P}_R^2$.

**Sylvester-Gallai Theorem:** Given a finite set of points in $\mathbb{P}_R^2$, there always exist a line that contains exactly two of the points or a line that contains all of the points.

**Problem 6.6** Using the dual projective plane and Sylvester-Gallai Theorem, prove that given a line arrangement in $\mathbb{P}_R^2$, there always exists a pair of lines that intersect in a point not contained in any other line of the line arrangement or all the lines pass through a common point.

This is not true for $\mathbb{P}_C^2$. Can you find a counterexample? How many lines do you need for a counterexample?