OLGA RADKO MATH CIRCLE: ADVANCED 3

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Spring Quarter - Worksheet 6: Transformations of the projective spaces

In the previous lectures, we studied projective lines $\mathbb{P}^1_{\mathbb{R}}$ and projective planes $\mathbb{P}^2_{\mathbb{R}}$ in a projective space $\mathbb{P}^3_{\mathbb{R}}$. In this worksheet, we will learn some applications of projective geometry. First, we will start by recalling concepts and solving preliminary problems.

The projective plane $\mathbb{P}^2_{\mathbb{R}}$ has points with coordinates $[x : y : z]$ and lines given by equations $ax + by + cz = 0$. The dual projective plane $\mathbb{P}^{2\vee}_{\mathbb{R}}$ will be the space parametrizing lines in $\mathbb{P}^2_{\mathbb{R}}$. More precisely, a line $ax + by + cz = 0$ in $\mathbb{P}^2_{\mathbb{R}}$, will correspond to the point $[a : b : c]$ in $\mathbb{P}^{2\vee}_{\mathbb{R}}$.

Problem 6.0 For the following lines in $\mathbb{P}^2_{\mathbb{R}}$: Write down the coordinates of the corresponding points in $\mathbb{P}^{2\vee}_{\mathbb{R}}$.
Draw the lines in $\mathbb{P}^2_{\mathbb{R}}$ and the corresponding points in $\mathbb{P}^{2\vee}_{\mathbb{R}}$.
In $\mathbb{P}^{2\vee}_{\mathbb{R}}$ identify which triples of points are colinear.

- $l_0 : \{x = 0\}$
- $l_1 : \{y = 0\}$
- $l_2 : \{z = 0\}$
- $l_3 : \{x + y = 0\}$

Problem 6.1 Show that three lines $a_0x + b_0y + c_0z = 0$, $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ intersect at a single point if and only if the corresponding points ($[a_0 : b_0 : c_0]$, $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$) of $\mathbb{P}^{2\vee}_{\mathbb{R}}$ are colinear.

What is the relation between the common point in $\mathbb{P}^2_{\mathbb{R}}$ and the common line in $\mathbb{P}^{2\vee}_{\mathbb{R}}$?

As we established before, we will use coordinates $[x : y : z]$ for $\mathbb{P}^2_{\mathbb{R}}$ and coordinates $[a : b : c]$ in $\mathbb{P}^{2\vee}_{\mathbb{R}}$ We can assign a correspondence between points $[x : y : z]$ in $\mathbb{P}^2_{\mathbb{R}}$ and lines $xa + yb + zc = 0$ in $\mathbb{P}^{2\vee}_{\mathbb{R}}$. Notice that the line $xa + yb + zc = 0$ has a, b and c as variables and x, y and z as constants.

Problem 6.2 For the following points in $\mathbb{P}^2_{\mathbb{R}}$. Wirte down the equations of the corresponding lines in $\mathbb{P}^{2\vee}_{\mathbb{R}}$.

Draw the points in $\mathbb{P}^2_{\mathbb{R}}$ and the corresponding lines in $\mathbb{P}^{2\vee}_{\mathbb{R}}$.

- $p_1 : [1 : 1 : 1]$
- $p_2 : [1 : 0 : 1]$
- $p_3 : [0 : 1 : 1]$
- $p_4 : [1 : 1 : 0]$

For a curve C with equation $f(x,y,z)=0$ in $\mathbb{P}^2_{\mathbb{R}}$, with a point $[x_0,y_0,z_0]$ inside the curve. The equation of the tangent line to C at the point p is:

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)x+\frac{\partial f}{\partial y}(x_0,y_0,z_0)y+\frac{\partial f}{\partial z}(x_0,y_0,z_0)z=0$$

Problem 6.3 For the following curves and points compute the tangent lines:

- $C:=\{x^2+y^2-z^2=0\}, p:=[0:1:1], q:=[3:4:5]$
- $C:=\{x=0\}$, any point.
- $C:=\{x^3+y^3+z^3=0\}, p:=[0:1:-1], q:[1:0:-1]$

Now we will define the dual of a general curve in $\mathbb{P}^2_{\mathbb{R}}$.

For each point p in a curve C in $\mathbb{P}^2_{\mathbb{R}}$, we can take the tangent line to C at p. These tangent lines correspond to points in $\mathbb{P}^{2\vee}_{\mathbb{R}}$. These points in $\mathbb{P}^{2\vee}_{\mathbb{R}}$ form a curve $C^\vee$, called the dual curve of C.

If C is a smooth curve of degree d, then the curve $C^\vee$ is a curve of degree $d(d-1)$

Problem 6.4 Using the fact that the following conics are smooth, compute the defining function of $C^\vee$:

- $C := \{x^2 + y^2 - z^2 = 0\}$
- $C := \{xz - y^2 = 0\}$
- $C := \{xy + yz + zx = 0\}$

Hint: You can use that five points define a conic.

Let C be a curve given by an equation of the form $Px^2 + Qy^2 + Rz^2 = 0$. The equation of the tangent line to $[x_0 : y_0 : z_0]$ is $2Px_0x + 2Qy_0y + 2Rz_0z = 0$. Hence the points in $C^\vee$ are of the form $[2Px_0 : 2Qy_0 : 2Rz_0] = [Px_0 : Qy_0 : Rz_0]$.

Since $[x_0 : y_0 : z_0]$ is in C, we have that $Px_0^2 + Qy_0^2 + Rz_0^2 = 0$. Therefore the points in $C^\vee$ $[a : b : c] = [Px_0 : Qy_0 : Rz_0]$ are of the form $a^2 = P^2x_0^2$, $b^2 = Q^2y_0^2$ and $c^2 = R^2z_0^2$.

Therefore the equation of $C^\vee$ is $\frac{1}{P}a^2 + \frac{1}{Q}b^2 + \frac{1}{R}c^2 = 0$. Remember that in $\mathbb{P}^{2\vee}_{\mathbb{R}}$, a, b and c are the variables.

Problem 6.5 Following the method for curves of the form $Px^2 + Qy^2 + Rz^2 = 0$, compute the equation of the curve $C^\vee$ for the following conics:

- $Pxy + Qz^2 = 0$
- $Pxy + Qyz + Rzx = 0$

Verify that these general formulas were achieved in the particular cases of Problem 6.4

A line arrangement in $\mathbb{P}^2_{\mathbb{R}}$ is simply a collection of lines in $\mathbb{P}^2_{\mathbb{R}}$.

Sylvester-Gallai Theorem: Given a finite set of points in $\mathbb{P}^2_{\mathbb{R}}$, there always exist a line that contains exactly two of the points or a line that contains all of the points.

Problem 6.6 Using the dual projective plane and Sylvester-Gallai Theorem, prove that given a line arrangement in $\mathbb{P}^2_{\mathbb{R}}$ there always exists a pair of lines that intersect in a point not contained in any other line of the line arrangement or all the lines pass through a common point.

This is not true for $\mathbb{P}^2_{\mathbb{C}}$. Can you find a counterexample? How many lines do you need for a counterexample?

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