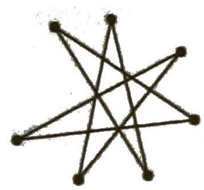
PLANAR AND DIRECTED GRAPHS

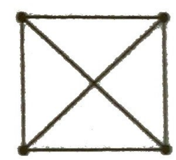
MATH CIRCLE (ADVANCED) 10/07/2012

We say that a graph is *planar* if it can be drawn in such a way that none of its edges intersect.

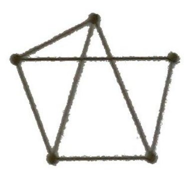
1) Prove that each of the graphs below is planar. That is, redraw them with no intersecting edges. In this case we say the graph is *properly depicted*.



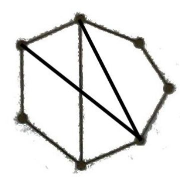
a)



b)



c)



d)

If a planar graph is properly depicted, it divides the plane into several regions called *faces*.

Given a graph, let V denote the number of vertices, E the number of edges, and F the number of faces (when properly depicted).

2) a) For each of the graphs in 1), calculate V, E, F?

b) What do you notice about $V - E + F$? Note this is referred to as Euler's Formula.

3) There are 7 lakes in Lakeland. They are connected by 10 canals so that one can swim through the canals from any lake to any other. How many islands are there in Lakeland?

4) a) Prove that for a planar graph, $2E \geq 3F$.

b) Prove that for a planar connected graph, if $V \geq 3$ then $E \leq 3V - 6$.

A graph in which each vertex is connected to every other vertex is called *complete*.

5) a) How many edges are there in a complete graph with $V = n$?

b) For what n is the complete graph on n vertices planar?

6) Prove that in any planar graph there exists a vertex with degree no more than 5.

7) Each edge of the complete graph with 11 vertices is colored either red or blue. We then look at the graph consisting of all the red edges, and the graph consisting of all the blue edges. Prove that at least one of these two graphs is not planar.

8) Prove that for a connected planar graph with no cycles of length ≤ 3, if $V \geq 3$ then $E \leq 2V - 4$.

9)* There are 20 points inside a square. They are connected by non-intersecting segments with each other and with the vertices of the square, in such a way that the square is dissected into triangles. How many triangles do we have?

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"