BÉZOUT'S THEOREM, PART 2

GLENN SUN
UCLA MATH CIRCLE ADVANCED 1

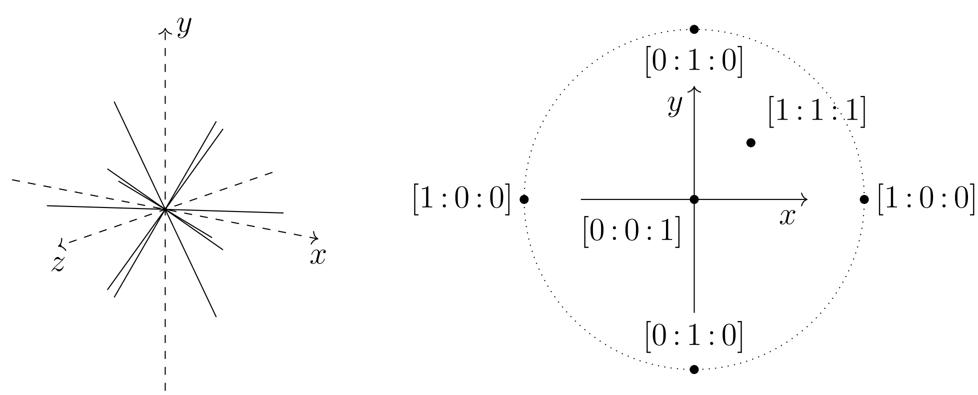
MAY 7, 2023

1 Recap: projective space $\mathbb{RP}^2$ and $\mathbb{CP}^2$

Last week, we started discussing algebraic geometry, the study of curves defined by polynomials, and took the first steps towards proving Bézout's theorem. This week, we will build on top of that to finish the proof. To remind you of the story so far:

- Bézout's theorem claims that if you "count the right way", there are always exactly mn solutions to the system of polynomial equations $f(x, y) = 0$, $g(x, y) = 0$, where $m = \deg(f)$ and $n = \deg(g)$. Equivalently, the zero sets of f and g have mn intersections.
- We started figuring out "the right way". First, f and g must not have any common factors. If they have common factors, then they will have infinitely many intersections.
- Lastly, we introduced the notion of the projective plane, a space where even "parallel" lines intersect.

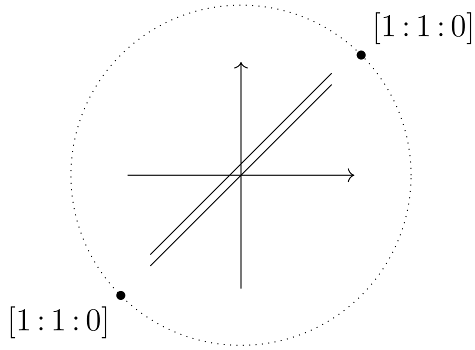
To remind you, we had two equivalent pictures of the projective plane:



The picture on the right is the picture on the left intersected with the plane $z = 1$. Every point inside the circle is labeled with a ratio representing the line that created it. Note that a ratio $[x : y : z]$ uniquely specifies a line through the origin (consisting of all the points that are in this ratio). For example, the point $[0 : 0 : 1]$ represents the z-axis.

We think about the circle in the picture on the right "at infinity", surrounding all of the other points. The points on the circle correspond to lines in the xy-plane in the left, which do not intersect $z = 1$. So the projective plane is just "the normal plane, plus a circle of infinities." The projective plane is called $\mathbb{RP}^2$, or $\mathbb{CP}^2$ if we allow complex numbers.

Let's get a little more intuition about why we call it "infinity". Below, I've plotted $y = x$ and $y = x + 1$, and then I zoomed out.



The lines never intersect on the normal plane, but if you had to say they intersect somewhere, it's pretty natural to create a new "infinite point" in the direction that both lines point, and say that they intersect there. This is exactly what happens in the projective plane: we have a new point at infinity for every direction, so they form a circle.

Now, let's investigate the details. Maybe you can see where two parallel lines should intersect, but can you predict the infinite point where $y = x^2$ and the y-axis intersect? What about the infinite point(s?) where two concentric circles intersect? We need to do some algebra!

We know that a curve is just a bunch of points stuck together, and that every point can be thought of as a line through the origin. The lines stuck together form a surface in 3D. What is the equation of the surface? Let me first pull it out of thin air: it is called the *homogenization*, and is given by the formula

$$\hat{f}(x, y, z) = z^n f(\tfrac{x}{z}, \tfrac{y}{z}),$$

where f has degree n. For example, if $f(x, y) = y - x - 1$, then $\hat{f}(x, y, z) = z(\frac{y}{z} - \frac{x}{z} - 1) = y - x - z$. To describe homogenization a different way, look at each term, and if it is "missing" degree, fill it up with z.

(Now, to check that homogenization makes sense, notice that once you intersect $\hat{f}$ with $z = 1$, you recover the original equation f as expected, and the surface is made up of many lines through the origin: whenever (x, y, z) is on the surface, so is every other point (kx, ky, kz) on the line, because every term has the same degree, and k^n factors out of the equation.)

Finally, to solve a system of equations over the projective plane, it's easy: first homogenize, then just solve the new system as you would any old system.

In our running example, $y = x$ is already homogenous, and we noted above that $y = x + 1$ homogenizes in $y = x + z$. These two equations imply $z = 0$, and we are free to choose x, after which $y = x$ is determined. So the solutions are all of the ratio $[1 : 1 : 0]$. We say that this projective point is the solution to the system of equations, as in the picture above.

Problem 1. (Same as Problem 11 from last week, skip if already did.) Allow yourself to use complex numbers. Find all projective solutions to the following system of equations. How many solutions are there over $\mathbb{R}^2$ vs $\mathbb{CP}^2$? Is the number predicted by Bézout's theorem (product of degrees)?

1. $y = x^2$ and $x = 1$.

2. $y = x^2 + 1$ and $y = 0$.

3. $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

2 Multiplicity

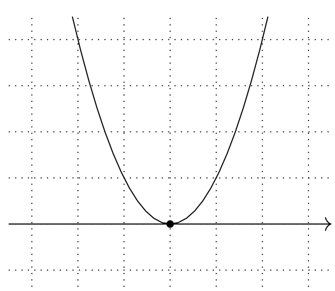
Alright! Let's finish the story now with the concept of multiplicity. We'll take a slight detour to consider the easier case of multiplicity in single variable polynomials, before going back to study multiplicity in the two-variable polynomials we've been looking at until now.

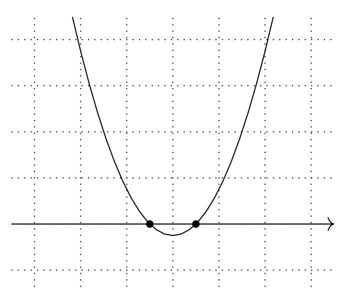
Given a single variable polynomial $f(x)$, we first factor it completely into linear factors (possibly with complex roots). For example, say we get $f(x) = (x-1)^2(x-3)(x-i)^3$. The roots are 1, 3, and i. The *multiplicity* of the root a is defined as the exponent of $(x-a)$, so 1 has multiplicity 2, 3 has multiplicity 1, and i has multiplicity 3.

Problem 2. Factor the following single variable polynomials completely over $\mathbb{C}$, and determine the multiplicity of the roots.

1. $x^3 - 2x^2$
2. $x^4 + 2x^2 + 1$

Now let's talk about the geometric meaning of multiplicity. If a root a has multiplicity m, it means that there exists a way to tweak the coefficients of the polynomial a little bit, so that the root splits into m different roots. For example, consider x^2, where the root 0 has multiplicity 2. The polynomial is really $x^2 + 0x + 0$, so if we slightly tweak the constant term into $x^2 - 0.0001$, the root 0 has now split into the two roots 0.01 and -0.01.



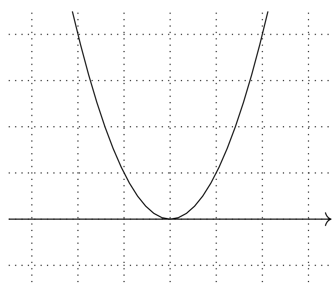


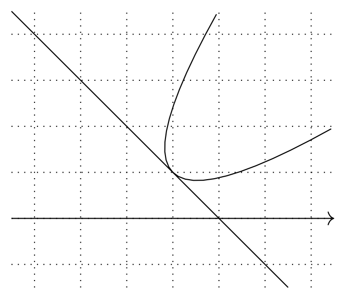
Note that not every tweak will split the root. For example, $1.001x^2$ still has 0 as its only root. So finding the right coefficient(s) to tweak will require some creativity!

Problem 3. (Challenge) Prove that the algebraic and geometric characterizations of multiplicity are the same.

Problem 4. Show geometrically that x^3 has 0 as a root of multiplicity 3. (That is, tweak the coefficients a little to reveal 3 distinct roots near 0.)

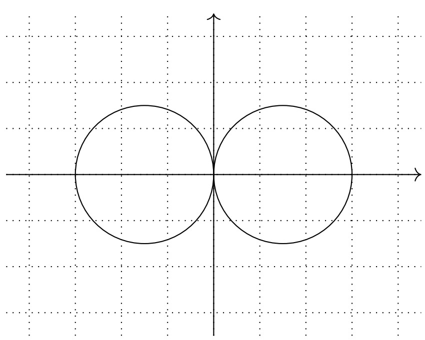
The reason we stress the geometric meaning is because the geometric meaning translates more naturally to the 2-variable cases. Since roots of a polynomial $f(x)$ are really just intersections between $g(x, y) = y - f(x) = 0$ and the x-axis $h(x, y) = y = 0$, we definitely want to say that the picture on the left depicts a root with multiplicity 2. And so when we rotate the entire picture, there is no reason the multiplicity should change. In the picture on the right, it's no longer very clear how to compute multiplicity algebraically, but the geometric meaning is still easy to verify visually: just translate the parabola a little.



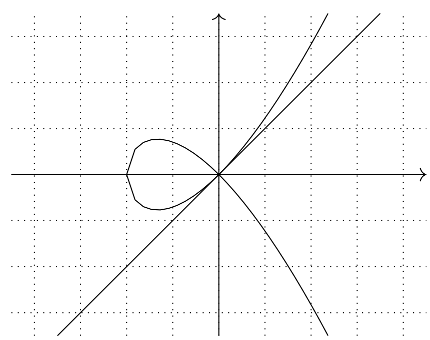


Problem 5. Determine the multiplicity of intersection in the following pictures:

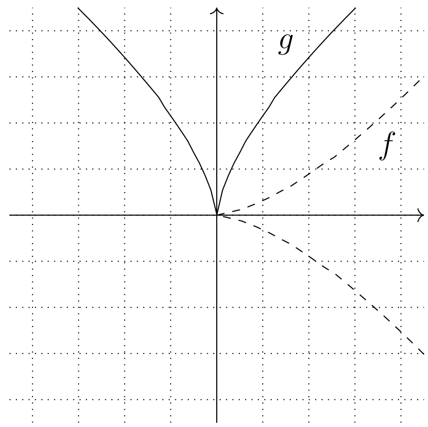
1. Two tangent circles



2. $y = x$ and $y^2 = x^2(x + 1)$ (The scale of the graph is 0.5 per box.)



Problem 6. Consider the curves $f(x, y) = y^2 - x^3$ and $g(x, y) = y^3 - x^2$. They are graphed below. (The scale of the graph is 0.1 per box.)



1. What is the multiplicity of intersection at $(0, 0)$?

2. Both f and g are polynomials of degree 3. We noted in the beginning that this should mean there are 9 total points of intersection if we "count the right way". You found 4 of them in part 1, now find the remaining 5 points of intersection.

For the following challenge problems, you may not be able to visualize the intersection multiplicity, but you can compute it algebraically, like our very first example.

To remind you of the method, if you wanted to compute the multiplicity of the root $(0,0)$ between the parabola $f(x,y) = y - x^2 = 0$ and the x-axis $g(x,y) = y = 0$, we could make a small shift to f by ϵ, making it $f^*(x,y) = y - x^2 + \epsilon$. (Figuring out where ϵ goes might take some creativity!) Solving this with $g(x,y) = y = 0$, we get $x^2 - \epsilon = 0$ and $x = \pm\sqrt{\epsilon}$. There are two solutions near $(0,0)$ now: $(0, \sqrt{\epsilon})$ and $(0, -\sqrt{\epsilon})$, and hence the multiplicity is 2.

Problem 7. (Challenge) In Problem 1, you found 2 intersection points for the two concentric circles $f(x,y) = x^2 + y^2 - 1$ and $g(x,y) = x^2 + y^2 - 4$. Now, find the multiplicity of those intersections to show that there are really 4 points of intersection.

Problem 8. (Challenge) Find all projective intersections with multiplicities between the curves $y^2 = x^3$ and $y^2 = 2x^3$.

3 Putting it together

Problem 9. Recall that Bézout's theorem says that if you solve over $\mathbb{CP}^2$ and count with multiplicities, and f and g don't share common factors, then they always intersect at exactly mn points, where $m = \deg(f)$ and $n = \deg(g)$. First, let's prove a special case: when f and g are homogeneous.

1. Let $f(x, y) = xy$ and $g(x, y) = (x - 1)(y - 1)$. Find all intersections of their zero sets over $\mathbb{CP}^2$ and the multiplicity of each intersection. (Graph them first!)

2. Show that if f and g are both unions of straight lines, i.e. products of degree 1 factors, then Bézout's theorem holds.

3. Two weeks ago, we proved the fundamental theorem of algebra for single-variable polynomials. It says that over $\mathbb{C}$, a single-variable polynomial can be always be factored into linear terms. Using this, prove that over the complex numbers, if $h(x, y)$ is homogeneous of degree k, then there exist $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ such that $h(x, y) = (s_1x - r_1y) \cdots (s_kx - r_ky)$, and the roots of h are $[r_1 : s_1], \ldots, [r_k : s_k]$. (Hint: homogenization!)

4. Using part 3, show that a polynomial $h(x, y)$ is a union of k straight lines if and only if it is homogeneous of degree k.

Problem 10. Now for the full theorem.

1. Let f and g be any polynomials now. Give a method to continuously change some coefficients in f and g so that they both become homogeneous.

2. Give a method to further continuously change the coefficients so that f and g end up sharing no common factors.

3. The roots of a polynomial vary continuously with its coefficients. Using this, prove Bézout's theorem.

4 Bonus: Computing multiplicity

The definition we gave for multiplicity was sufficient for proving Bézout's theorem, but as you saw, you have to "see" the right perturbation necessary to reveal the intersections, and it's very hard to be sure that no intersections are hiding in $\mathbb{C}$. Here, we present one method to compute multiplicity algorithmically.

The *resultant* of two polynomials $f(z) = a_n z^n + \cdots + a_1 z + a_0$ and $g(z) = b_m z^m + \cdots + b_1 z + b_0$ is the unique $\mathbb{C}$-valued function $\operatorname{res}(f, g)$ satisfying:

1. $\operatorname{res}(a_0, g) = a_0^m$ and $\operatorname{res}(f, b_0) = b_0^n$.
2. $\operatorname{res}(z - a_0, z - b_0) = a_0 - b_0$.
3. $\operatorname{res}(g, f) = (-1)^{mn} \operatorname{res}(f, g)$.
4. $\operatorname{res}(fg, h) = \operatorname{res}(f, h) \operatorname{res}(g, h)$.

Problem 11. Using the above properties, compute the resultants below.

1. $\operatorname{res}(z^2 - 1, z - 3)$

2. $\operatorname{res}(z + 3, 2z)$

Problem 12. Describe a general strategy to compute the resultant of any two polynomials.

Problem 13. Let's investigate what the resultant means.

1. Show that
$$\operatorname{res}(f,g) = a_n^m b_m^n \prod_{i,j} (r_i - s_j),$$
where $r_1, \ldots, r_n$ and $s_1, \ldots, s_m$ are the roots of f and g counted with multiplicity, respectively. (Hint: check that this formula satisfies the four defining criteria of $\operatorname{res}(f,g)$. Why is that enough for the proof?)

2. Using the first part, show that $\operatorname{res}(f,g) = 0$ iff f and g have a common root.

Now let $f(x,y,z)$ and $g(x,y,z)$ be homogeneous polynomials of degrees n and m respectively. If we consider f and g as polynomials in z (with coefficients being polynomials in x and y), the resultant $\operatorname{res}(f,g)$ is a polynomial in x and y.

Problem 14. Homogenize $y = 1$ and $(x-2)^2 + y^2 = 1$. Then compute their resultant. Compute their points of intersection (with multiplicities) as in previous sections, then compare with the resultant: what do you see? Make a conjecture.

Problem 15. We will now use resultants to figure out the multiplicity of intersections. Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous of degree n and m respectively, and we always take the resultant over z.

1. Show that $\operatorname{res}(f, g)$ is a homogeneous polynomial of degree nm.

2. Show that if $[a : b : c]$ is a common root of f and g, then $bx - ay$ is a factor of $\operatorname{res}(f, g)$.

3. Show that
$$\operatorname{res}(f, g) = \prod_{[a:b:c]} (bx - ay),$$
where the product is over common roots $[a : b : c]$ of f and g, listed with multiplicity. (Now, this is the perturbation-based multiplicity we defined earlier.)

Problem 16. Try to use this method to compute the multiplicity of intersections for the following example: $x = 0$ and $y = x^2$. What problems do you run into? How can you fix it?

(Hint: when we chose to interpret the $\mathbb{CP}^2$ as the intersection of the lines through the origin with the plane $z = 1$, there was nothing really special about $z = 1$, apart from making our formulas easy. What if you chose a different plane?)

Problem 17. (Challenge) Recall from lasst quarter that Gaussian elimination is an algorithm on matrices by performing the following row operations: (1) switch two rows, (2) multiply a row by a non-zero constant, (3) add some multiple of one row to another.

Let $f(z) = a_3z^3 + a_2z^2 + a_1z + a_0$ and $g(z) = b_2z^2 + b_1z + b_0$. Show how one can compute $\operatorname{res}(f, g)$ by performing Gaussian elimination on the following matrix.

$$\begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 \\ a_1 & a_0 & b_1 & b_0 & 0 \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & 0 & b_2 & b_1 \\ 0 & a_3 & 0 & 0 & b_2 \end{bmatrix}$$

This method for obtaining a number (here, the resultant) from a matrix is very common, and is called the *determinant* of the matrix. Ask your instructor for details if you are interested.