ORMC AMC 10/12 Training Week 6
Diophantine Equations

May 7, 2023

1 Warm-up: General Diophantine Equations

1. Solve the equation
$$6x + 10y - 15z = 1.$$

2. Prove that the equation $x^5 - y^2 = 4$ has no solutions in integers.

3. Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

4. Solve in the integers the Diophantine equation $x^4 - 6x^2 + 1 = 7 \cdot 2^y$.

5. Determine all triples (x, y, z) of integers satisfying the equation
$$3x + 4y + 5z = 6.$$

6. Let n be a positive integer. Suppose that there are 666 ordered triples (x, y, z) of positive integers satisfying the equation
$$x + 8y + 8z = n.$$
Find the maximum value of n.

2 Pell Equations

Pell Equations are a specific class of diophantine equations of the form $x^2 + ny^2 = 1$.

Theorem 1. The equation $x^2 - ny^2 = 1$ has a non-trivial solution if and only if n is not a square. If n is not a square, then infinite solutions can be generated by the *fundamental solution* (x_1, y_1), the solution with the smallest value of x and y.

2.1 Proof (Exercises)

1. Show that if n is a square then the equation $x^2 - ny^2 = 1$ has only the trivial solution $x = 1$ and $y = 0$.

2. Assume henceforth that n is a non-square. Show that if (x, y) is a solution to $x^2 - ny^2 = 1$, then the fraction $\frac{x}{y}$ is a close approximation for $\sqrt{n}$.

3. Let $\frac{h_i}{k_i}$ denote the i-th convergent of the continued fraction representation for $\sqrt{n}$. Then there exists an $i \in \mathbb{N}$ such that $x_1 = h_i$ and $y_1 = k_i$. That is, the fundamental solution can always be found by testing convergents until a solution is found. The proof for this statement is out of the scope for this lecture but will be assumed to be true. Using this method, find the fundamental solutions for $n = 2$ and $n = 3$.

4. Show that if (x, y) and (a, b) are two solutions to Pell's equation, then $(xa + nyb, xb + ya)$ is a third solution.

5. For all $k > 1$, define $(x_k, y_k) := x_k + y_k\sqrt{n} = (x_1 + y_1\sqrt{n})^k$. Show that (x_k, y_k) is a solution to $x^2 - ny^2 = 1$. We say that (x_k, y_k) is *generated* by (x_1, y_1).

6. Show that all solutions are generated by the fundamental solution (x_1, y_1).

2.2 Examples

1. Find the 3 smallest solutions to $x^2 - 2y^2 = 1$.

2. **(AMC 12)** A *triangular number* is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

2.3 Exercises

1. **(ARML)** Let n be a positive integer, and consider the list $1, 2, 2, 3, 3, 3, ..., n, n, ..., n$ where the integer k appears k times in the list for $1 \leq k \leq n$. The integer n will be called "ARMLy" if the median of the list is not an integer. The least ARMLy integer is 3. Compute the least ARMLy integer greater than 3.

2. Prove that if n is a natural number and $(3n + 1)$ and $(4n + 1)$ are both perfect squares, then 56 will divide n.

3. **(AIME)** Find the largest integer n satisfying the following conditions:
(i) n^2 can be expressed as the difference of two consecutive cubes;
(ii) $2n + 79$ is a perfect square.

4. **(British Math Olympiad)** Find the first integer $n > 1$ such that the average of $1^2, 2^2, \ldots, n^2$ is itself a perfect square.

5. **(European Girls Math Olympiad)** Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2 + 1, n^2 + 2, \ldots, n^2 + 2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of S of the form $7m + 3$ or $7m + 4$, where m is an integer.

3 Advanced Problems

(General Number Theory)

1. **(China 2017)** Let $m \geq n > 1$ be integers. Let $a_1, a_2, \ldots, a_n$ be n distinct and relatively prime numbers not exceeding m. Show that for any real x, there exists an i for which

$$||a_i x|| \geq \frac{2}{m(m+1)}||x||$$

where $||x||$ denotes the distance between x and the nearest integer to x .

2. **(IMO 2017)** An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n = 1.$$

3. **(Putnam 2022)** Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$ for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 (mod 5).

4. **(USAMO 2023)** Let $n \geq 3$ be an integer. We say that an arrangement of the numbers $1, 2, \cdots, n^2$ in a $n \times n$ table is row-valid if the numbers in each row can be permuted to form an arithmetic progression, and column-valid if the numbers in each column can be permuted to form an arithmetic progression. For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?