Graphs Crossings on Surfaces

Last packet we began studying the properties of the drawings of a graph. Remember that when we draw a graph the vertices are drawn as points, and the edges are drawn as lines connecting the points. Of course, the lines representing edges will intersect at the points representing vertices. However, sometimes the lines also intersect at points that do not represent vertices, and we call these extra intersections **crossings**. Sometimes we can draw a graph in different ways, so that the different drawings have different numbers of crossings. The **crossing number** of a graph G is the smallest number of crossings that any drawing of G must always have.

Problem 1. *The drawing of the graph G below has five crossings.*

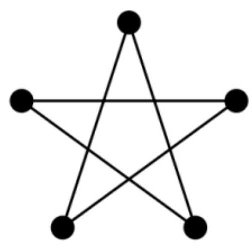


Figure 1: Graph G with five crossings

However, by moving the vertices around, we can redraw G in a way that has no crossings.

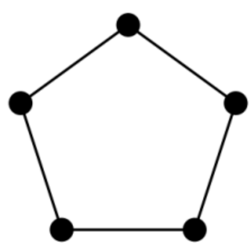


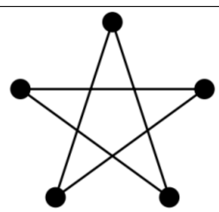
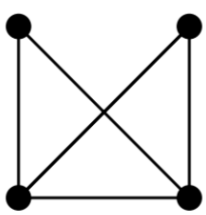
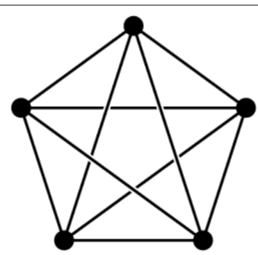
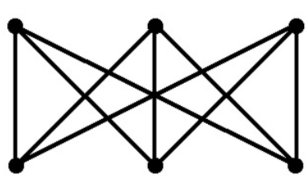
Figure 2: Graph G with no crossings

(i) *The crossing number of G is equal to __________.*

(ii) *The drawing in Figure 1 and the drawing in Figure 2 are isomorphic, meaning they represent the same graph G. Label the vertices and edges in Figure 1 and Figure 2 so that corresponding vertices/edges in the drawings have the same labels.*

So far we have only been drawing graphs on the two-dimensional plane. But, why can't we draw on other two-dimensional surfaces, like the sphere, the cylinder, or the Möbius strip? Well, it turns out we can! Let S represent your favorite one of these surfaces and let G represent your favorite graph. If we draw G on S, the lines representing edges might accidentally intersect at points that do not represent vertices, and we call these extra intersections **S-crossings**. The **S-crossing number** of G is the smallest number of S-crossings that any drawing of G on S must always have.

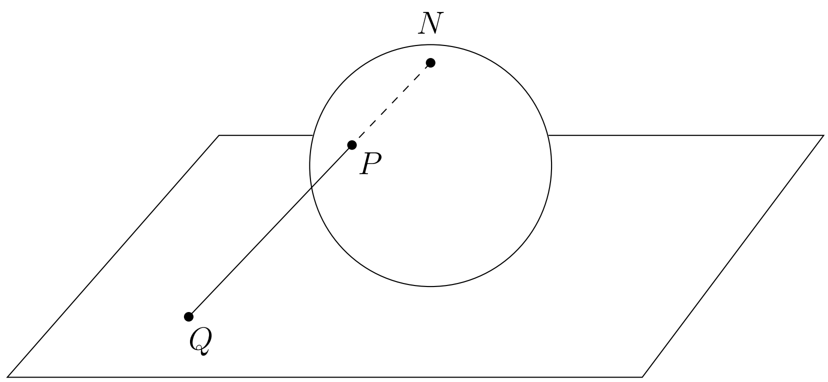
Problem 2. *Try drawing each of the graphs on the surfaces that the instructor has given you. Then, wrtie down what you think the crossing number of each graph on each surface is, in the chart below.*

	Plane	*Sphere*	*Cylinder*	*Möbius Strip*
				
				
				
				

After experimenting with drawings on different surfaces, you would be right to wonder why some surfaces help us remove more crossings than we can on the plane, while some aren't any better than the plane. Fully answering this question leads to a lot of beautiful and advanced mathematics. Today we will just scratch the surface (hahaha get the pun?).

Understanding the sphere

Imagine you put a hollow, translucent sphere on top of a table and shined a light at the "north pole". Then each point on the sphere has its shadow on the table. The shadow Q, the point on the sphere P, and the north pole N, all lie on the same line, as in the diagram below.



This way of associating the points on the sphere to points (their shadows) on the plane is called **stereographic projection**. We can visualize it better through the following video.

Stereographic projection
`https://www.youtube.com/watch?v=VX-0Laeczgk`

Problem 3. *Stereographic projection associates circles that don't pass through the north pole to circles on the plane. Can you figure out what happens to circles that pass through the north pole?*

Stereographic projection is a very useful tool for us because it lets us associate drawings of graphs on the sphere to drawings of graphs on the plane.

Problem 4. *Consider the graph below that is drawn on the sphere, which is called the* ***spherical tetrahedron****.*

(i) *Draw what the spherical projection of the graph onto the plane looks like if the north pole coincides with one of the vertices of the original graph.*

(ii) *Draw what the spherical projection of the graph onto the plane looks like if the north pole lies on one of the edges of the original graph (but not on a vertex).*

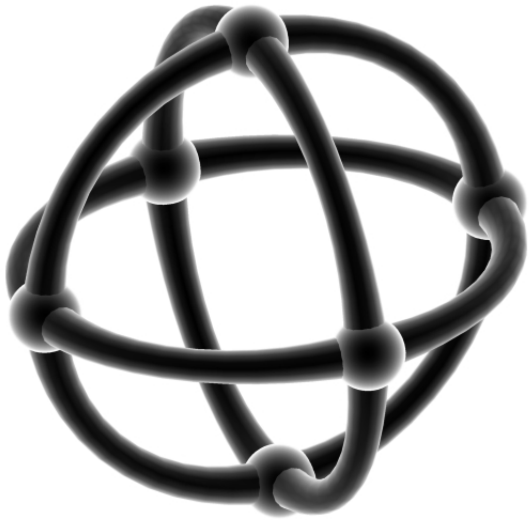
This problem continues to the next page.

This is a continuation of the problem on the prior page.

(iii) *Draw what the spherical projection of the graph onto the plane looks like if the north pole does not coincide with a vertex or lie on one of the edges of the graph.*

(iv) *Your answer to part* ***(iii)*** *should be a planar graph. If needed, redraw it below so that there are no crossings. Next, compute the Euler characteristic of this planar graph.*

Problem 5. *Consider the graph below that is drawn on the sphere, which is called the **spherical octahedron**.*



(i) *Draw what the spherical projection of the graph onto the plane looks like if the north pole coincides with one of the vertices of the original graph.*

(ii) *Draw what the spherical projection of the graph onto the plane looks like if the north pole lies on one of the edges of the original graph (but not on a vertex).*

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(iii) *Draw what the spherical projection of the graph onto the plane looks like if the north pole does not coincide with a vertex or lie on one of the edges of the graph.*

(iv) *Your answer to part **(iii)** should be a planar graph. If needed, redraw it below so that there are no crossings. Next, compute the Euler characteristic of this planar graph.*

Problem 6. *What do you notice about you answer to Problem 4 part (iv) and Problem 5 part (iv)?*

Problem 4 and Problem 5 give us examples of a much more important phenomenon. Suppose you draw a graph on the sphere that has zero sphere-crossings. Then the stereographic projection of this graph (where the north pole is not on an edge or vertex) is a drawing on the plane that has zero planar-crossings! The opposite is also true; if you start with a graph drawn on the plane with zero plane-crossings, then it is the shadow of a drawing of the same graph with zero sphere-crossings. In other words, stereographic projection gives us a correspondence between drawings with zero sphere-crossings and drawings with zero-plane crossings.

This observation explains why your answers for the plane and sphere to Problem 2 are the same. It also explains your answer to Problem 6, since we learned last time that the Euler characteristic of any planar graph equals two. We can even generalize Problem 6 a bit further.

A **polyhedron** is a 3-dimensional shape that has flat polygonal faces, straight edges, and sharp corners. Some examples are:

Figure 3: A tetrahedron

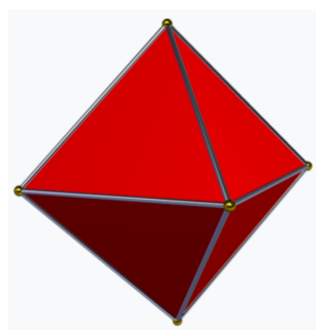

Figure 4: An octahedron

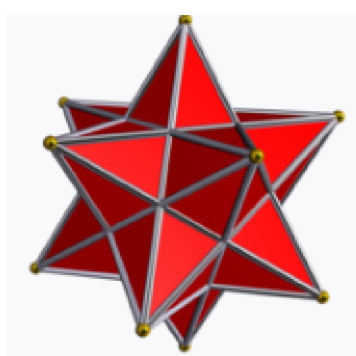

Figure 5: A small stellated dodecahedron

For a polyhedron P, the **Euler characteristic of** P (written χ) is defined to be

$$\chi = V - E + F.$$

where V is the number of vertices of P, E be the number of edges of P, and F be the number of faces of P. For the tetrahedron (see Figure 3), we have that

$$V = 4, \qquad E = 6, \qquad F = 4, \qquad \chi = 4 - 6 + 4 = \underline{\hspace{2cm}}.$$

For the octahedron (see Figure 4), we have that

$$V = 6, \qquad E = 12, \qquad F = 8, \qquad \chi = 6 - 12 + 8 = \underline{\hspace{2cm}}.$$

For the small stellated dodecahedron (see Figure 5), we have that

$$V = 12, \qquad E = 30, \qquad F = 12, \qquad \chi = 12 - 30 + 12 = \underline{\hspace{2cm}}.$$

Notice how the Euler characteristic of the small stellated polyhedron is different than that of the tetrahedron and octahedron. This is because the tetrahedron and octahedron are **spherical polyhedra**, meaning they are polyhedra that fit snuggly inside a sphere with all of the vertices touching the sphere. Meanwhile the small stellated dodecahedron is not spherical.

Spherical polyhedra are special because they can be represented as graphs on the sphere with zero sphere-crossings! For example, the tetrahedron in Figure 3 corresponds to the drawing of the graph in Problem 4 on the sphere. Moreover, the octahedron in Figure 4 corresponds to the drawing of the graph in Problem 5 on the sphere. These graph representations of spherical polyhedra let us prove the following result.

Problem 7. *Prove that the Euler characteristic of any spherical polyhedron is equal to two.*

Problem 7 sheds light (haha get it?) on the somewhat mysterious infinite face of a planar graph. Recall that when we defined Euler characteristic for planar graphs, we asked you to include the exterior infinite face in the face count. This is because a planar graph is the shadow of a drawing on the sphere that is the graph of a spherical polyhedra. The infinite face of the planar graph becomes a finite face of the polyhedron, just like all the other faces!

This is the end of our journey exploring graphs on spheres. The key insight is that we have a tool that lets us switch between graphs on the plane and graphs on the sphere: spherical projection! Can you think of a similar way to associate points on the cylinder to points on the plane? What about ways of associating points on the Möbius strip to points on the plane? The complete answers to these questions require the rich and beautiful math of topology, which we hope to inspire you all to learn someday.