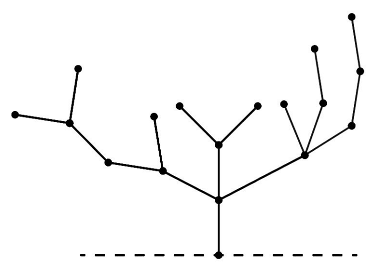
HACKENBUSH

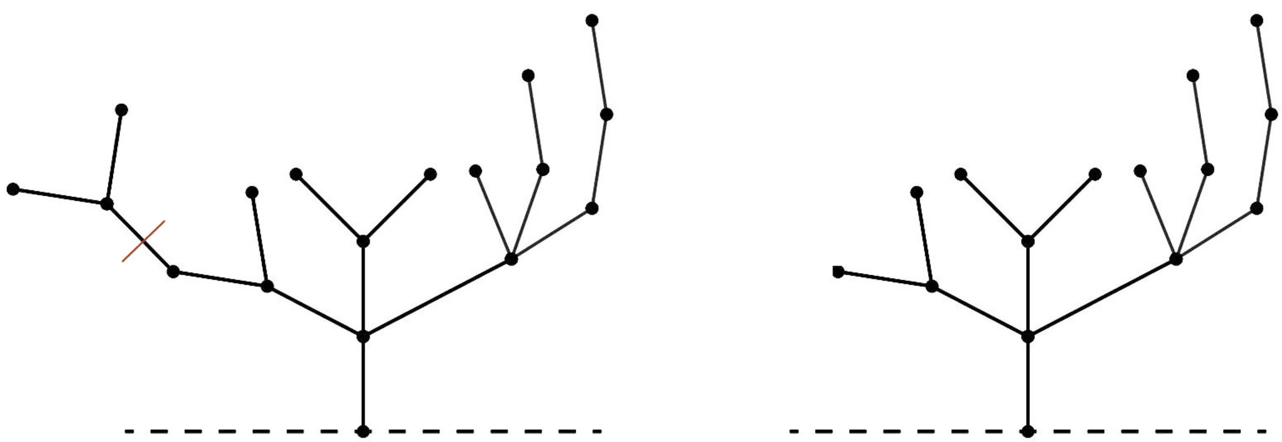
MAX STEINBERG FOR OLGA RADKO MATH CIRCLE
INTERMEDIATE 2

1. Hackenbush

Let's play a game. Consider the following stick diagramme. On your turn, you can cross out any line (except the dashed line that marks the ground). Then, any line that is no longer connected to the ground falls off.

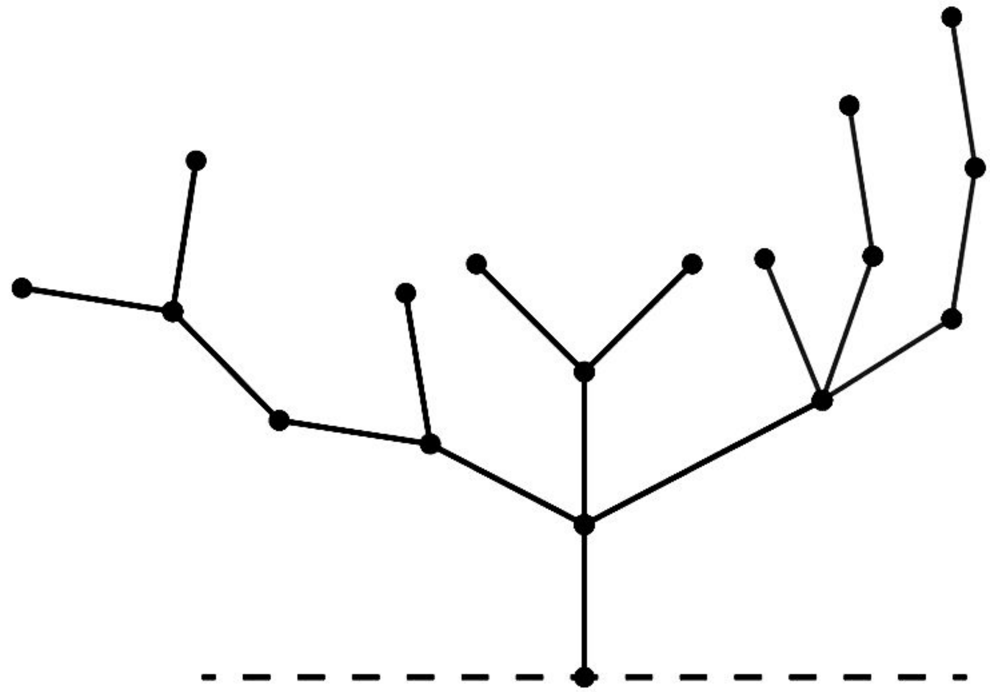


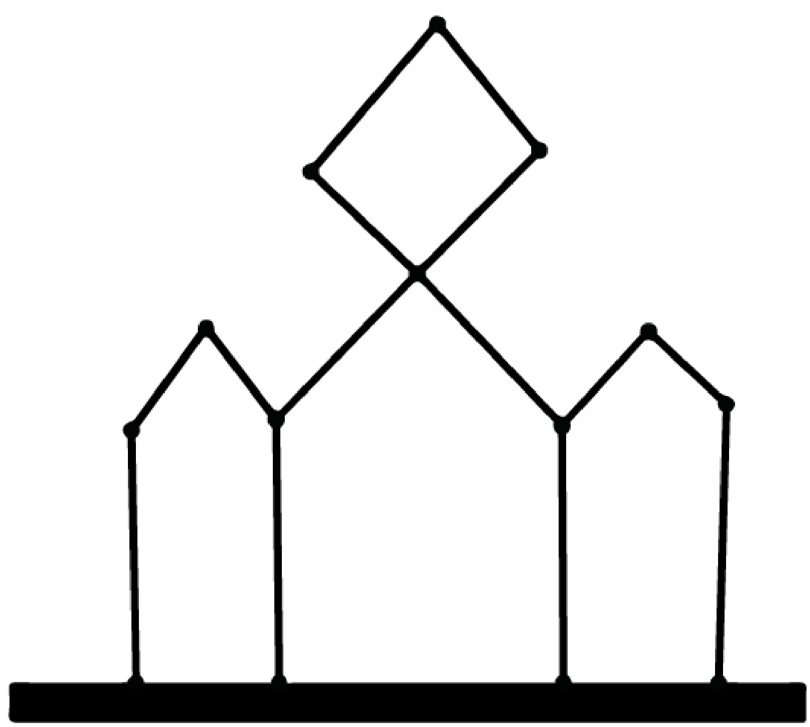
Let's look at an example move:



I made a cut shown above, and that line was removed. Then, there were two more lines not connected to the ground, so they fell off as well.

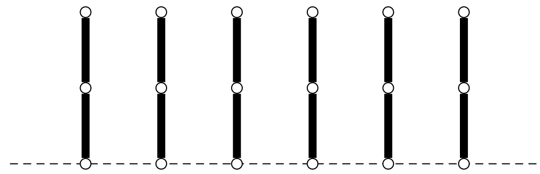
Problem 1. Play Hackenbush with the students sitting next to you. Try to find a strategy to win the game (a strategy may include selecting whether or not you wish to go first). Try playing on a variety of trees (some examples are given on the next page).





2. How to Win Hackenbush

Let's first consider a very simple example. We are playing Hackenbush and it is your turn. The current tree is a *bamboo garden* (pictured below):



Problem 2. What are all the legal moves for you to make?

Solution. Cut 1 or 2 segments from any bamboo stalk (vertical tree).

Problem 3. Can you identify another game you know about with the same set of legal moves?

Solution. 6 pile Nim with piles $2, 2, 2, 2, 2, 2$.

Problem 4. Who wins that game? Can you determine who would win our Hackenbush game from this position?

Solution. The Nim sum $2 \oplus 2 \oplus 2 \oplus 2 \oplus 2 \oplus 2 = 0$ so Player 2 wins.

Problem 5. How can you determine who wins a game of Hackenbush that starts from a bamboo garden?

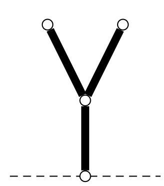
Solution. Count the number of segments in each bamboo stalk and take their Nim sum. Player 2 wins if the result is 0 and Player 1 wins otherwise. It is very clear that this is just a game of Nim.

Using the ideas from the past few problems, we will assign an **evaluation** to a Hackenbush game, very similar to how we assigned a number to a Nim game. For a single bamboo stalk, the evaluation is the number of segments in the stalk. In a more complicated tree, our evaluation may be more difficult.

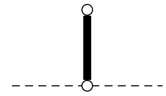
Problem 6. How can we assign an evaluation to a bamboo garden?

Solution. It is just the Nim sum of each stalk.

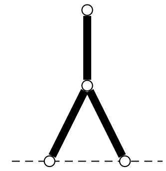
Problem 7. Consider a slightly more complicated type of tree (pictured below). How can we evaluate Hackenbush played on this tree? Why? (Feel free to play the game on the tree below, and trees similar to it, to get a feel for the strategy.)
Hint: consider a **reduction argument**: try to reduce a complicated problem to a simpler case.



Solution. This is the **colon principle**: when two branches meet, we replace the two branches by a single branch with a number of segments equal to the Nim sum of the lengths of the branches. So in this case, we have two branches that meet with lengths 1 and 1, so we replace the join with a branch with $1 \oplus 1 = 0$ segments:



There is one additional type of complication we can have in Hackenbush trees.

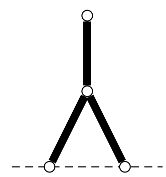


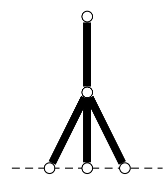
So far, we have learned how to evaluate Hackenbush trees that are bamboo stalks or that split off into multiple branches. But what can we do to evaluate branches that join again, such as in the above tree?

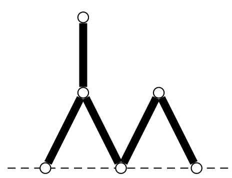
Problem 8. Who wins Hackenbush played on the above tree?

Solution. Player 1 will win. Cut the top branch, which leaves two moves. Whatever Player 2 does, Player 1 cuts the other branch and wins.

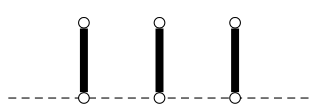
Problem 9. For each tree below, write a list of all legal moves you can make. Draw a tree without joins that has the same set of legal moves. Evaluate each of those trees.

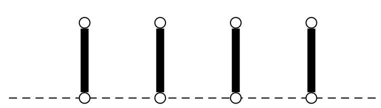


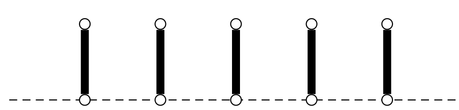




Solution. Only the trees are drawn since the legal moves are obvious for the trees below:





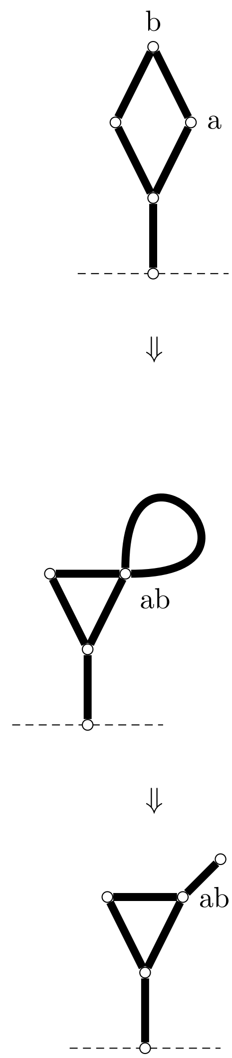


Evaluations are $1, 0, 1$ respectively.

Problem 10. For each of the trees in the previous problem, determine who wins Hackenbush played on those trees. Does this align with the evaluations you obtained?

Solution. Player 1, Player 2, Player 1, which does align.

Problem 11. The **fusion principle** states that given a loop in a Hackenbush tree, we can "fuse" two nodes together in the loop. Explain, in your own words, why this is true in the following example (where we fuse a and b):



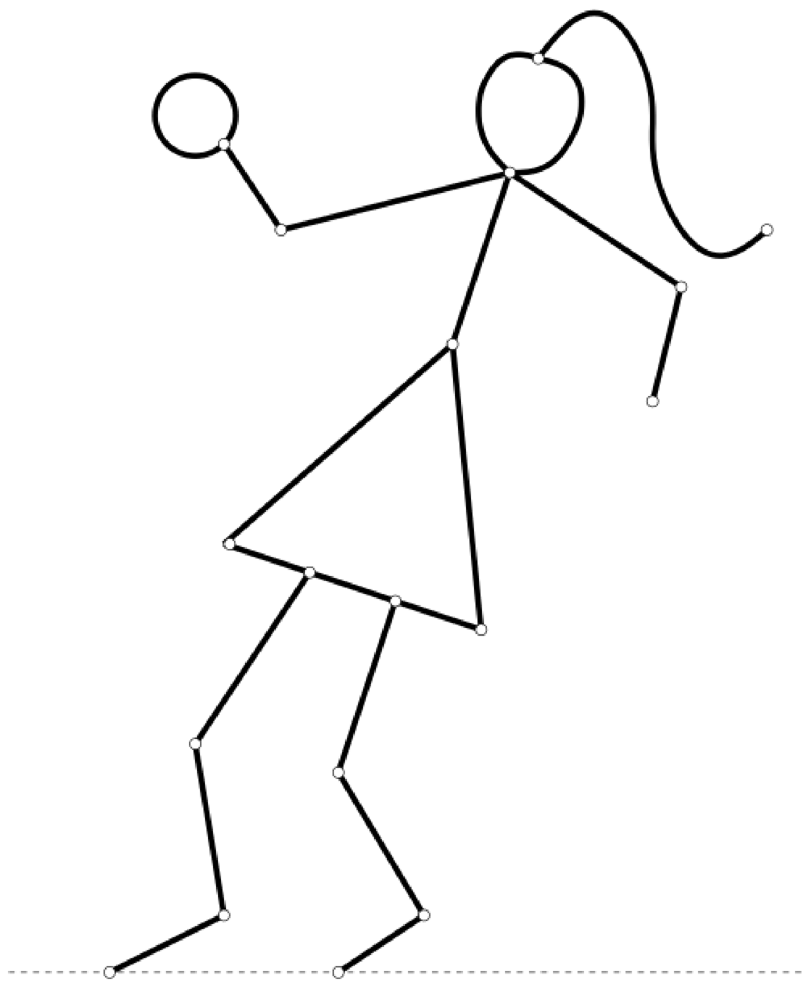
Solution. Solutions may vary, just make sure what they say makes sense.

The fusion principle is rather difficult to prove in general, so you can assume it to be true for the rest of the packet.

Problem 12. Evaluate the trees on Page 2.

Solution. First tree: 4
Second tree: 0

Problem 13. Evaluate the following tree.



Solution. I think this is 2 but honestly I am not quite sure.

3. Difficult Problems

If you are finished doing all the above, but there still remains some time...

Problem 14. Consider *Red-Blue Hackenbush*. One player is the Red player and one player is the Blue player. Players may only cut branches matching their colour. How can we evaluate Red-Blue Hackenbush? Try playing this game with other students who have finished the rest of the packet.