Linear Algebra. Part 2

Prepared by Mark and Nikita on May 4, 2023

Problem 1:
Apple and banana together weigh 2 pounds, apple and coconut weigh 3 pounds, and banana and coconut weigh 4 pounds. How much do they weigh separately?

Problem 2:
Solve the system

$$\begin{aligned}
x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 &= 1,\\
x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 &= 2,\\
x_1 + 3x_2 + 5x_3 + 6x_4 + 6x_5 &= 3,\\
x_1 + 3x_2 + 5x_3 + 7x_4 + 8x_5 &= 4,\\
x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 &= 5.
\end{aligned}$$

Problem 3:
Solve

$$\begin{aligned}
5732x + 2134y + 2134z &= 7866,\\
2134x + 5732y + 2134z &= 670,\\
2134x + 2134y + 5732z &= 11464.
\end{aligned}$$

Problem 4:
On the board, there is a row of numbers and stars: 5, *, *, *, *, *, *, 8. Replace the stars with numbers so that the sum of every three numbers in a row is 20.

Problem 5:
Consider a tetrahedron where all faces have equal perimeters. Prove that the faces themselves are equal in shape and size.
Hint: To prove the equality of the faces, set up a system of equations representing the lengths of the edges. Deduce from this system that the pairs of opposite edges of the tetrahedron are of equal length.

Problem 6:
Ten people are seated at a round table, and a total of ten dollars is to be distributed among them. The rule states that each person should receive half the sum of the money his two neighbors received combined. Does this rule clearly determine a unique distribution of the money?

Problem 7:
There are seven coins in a row in some order (one each with weights 1, 2, ..., 7 grams). For each coin (except for the extreme ones), the sum of the weights of its neighbors is known.
What is the largest number of coins that can be guaranteed to know the weight?
Clue: Make sure you can express the weight of the second, fourth, and sixth coins in terms of known weights.

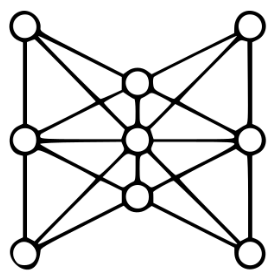
Problem 8:
In a 3×3 table, there are numbers filled in three of the cells (see figure). The objective is to fill in the remaining cells with numbers in such a way that the sums of the numbers in all rows, columns, and main diagonals are equal. Prove that there exists only one way to achieve this and proceed to complete the table accordingly.

1		3
5		

Problem 9:
An $m \times n$ table is filled by numbers. It turned out that for any two rows and any two columns, the sum of the numbers at the two opposite vertices of the formed rectangle is equal to the sum of the numbers at its other two vertices. However, some of the numbers have been erased, leaving only a portion of the table. Prove that at least $(n + m–1)$ numbers must remain in the table.

Problem 10:
A line of recruits stood facing the sergeant. On the command "Left turn" some turned to the left, while the rest turned to the right. It turned out that six times as many soldiers looked at the back of their neighbor's head as looked at their face. Then, at the command "About turn" everyone turned in the opposite direction. Now, there were seven times as many soldiers looking at the back of their neighbor's head as there were looking at their face. How many soldiers are in the line?

Problem 11:
There are 9 apples arranged on a table, creating 10 distinct rows with 3 apples each, as illustrated below:



It is given that the weight of nine rows is identical, while the tenth row has a different weight. You have access to electronic scales that can measure the weight of any group of apples for a cost of one dollar per weighing. Determine the minimum number of dollars you need to spend to identify the row with the differing weight.

Problem 12:

1. A group of robbers has stolen a bag of coins from a merchant. Each coin has an integer value in groshes. It is found that regardless of which coin is removed, the remaining coins can be distributed among the robbers so that each robber receives an equal sum in groshes. Prove that when one coin is set aside, the total number of coins is divisible by the number of robbers.
2. Consider the same scenario as in the first problem, but now the coins can have any positive value. Prove that the divisibility condition still holds.