OLGA RADKO MATH CIRCLE: ADVANCED 3

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Spring Quarter - Worksheet 3: Applications of Projective Geometry I

In the previous lectures, we studied projective lines $\mathbb{P}^1_{\mathbb{R}}$ and projective planes $\mathbb{P}^2_{\mathbb{R}}$ in a projective space $\mathbb{P}^3_{\mathbb{R}}$. In this worksheet, we will learn some applications of projective geometry. First, we will start by recalling some concepts and solving some preliminary problems.

Problem 1.0: Consider the projective line ℓ in the projective space $\mathbb{P}^2_{\mathbb{R}}$ parametrized by

$$[s+t : s-t : s+t].$$

Find a single linear equation that describes the line ℓ.
Solution 1.0:

Problem 1.1: Consider the projective plane P in $\mathbb{P}^3_{\mathbb{R}}$ parametrized by the equations:

$$[s - t : s + t + r : s - r : s + t + r].$$

Find a single linear equation that describes the plane P.
Solution 1.1:

Problem 1.2: Consider the point $p_0 := [1 : 0 : 0 : 0]$ in the projective space $\mathbb{P}^3_{\mathbb{R}}$. Consider the plane

$$P := \{[x_0 : x_1 : x_2 : x_3] \mid x_0 + x_1 + x_2 + x_3 = 0\}.$$

Describe the set of projective lines ℓ that pass through p_0 and intersect the plane P once.

Solution 1.2:

Problem 1.3: Let $r \in \mathbb{R}$ be a real parameter. For each r, we consider a projective line

$$\ell_r := \{[s+t : s-t : rs+rt] \mid [s:t] \in \mathbb{P}^1_{\mathbb{R}}\} \subset \mathbb{P}^2_{\mathbb{R}}.$$

Describe these lines for the values $r = 0, r = 1$, and $r = 2$.

Draw how these three projective lines ℓ_0, ℓ_1, and ℓ_2 intersect in the projective plane.

Is there a point in the projective plane that is contained in all the lines ℓ_r?

Solution 1.3:

A *family of lines* in $\mathbb{P}^2_{\mathbb{R}}$ is a set of lines $\{\ell_t \mid t \in \mathbb{R}\}$ parametrized with a real parameter t.

For instance, the lines:

$$\ell_t := \{[x_0 : x_1 : x_2] \mid tx_0 + x_1 + x_2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$$

is a family of projective lines in the projective plane. From the description, we can see that

$$\ell_0 \cap \ell_1 = [0 : 1 : -1].$$

We say that the family of lines *pass through the point* p_0 if every line ℓ_t in the family contains p_0. We may also call p_0 a *fixed point* of the family.

Problem 1.4: Find a family of projective lines in $\mathbb{P}^2_{\mathbb{R}}$ so that all of them contain the point $[0 : 1 : 0]$.

Can you find a family of lines ℓ_t such that they do not have a common intersection point?

Find a family of projective lines in the projective plane $\mathbb{P}^2_{\mathbb{R}}$ with a fixed point at infinity. Write the corresponding family of lines in $\mathbb{R}^2$.

Solution 1.4:

A *family of planes* in $\mathbb{P}^3_{\mathbb{R}}$ is a set of planes $\{P_t \mid t \in \mathbb{R}\}$ parametrized with a real parameter t.

For instance, the planes:

$$P_t := \{[x_0 : x_1 : x_2 : x_3] \mid x_0 + tx_1 + tx_2 + x_3 = 0\} \subset \mathbb{P}^3_{\mathbb{R}}$$

is a family of projective planes in the projective space. From the description, we can see that

$$P_0 \cap P_1 = \ell := \{[t : s : -s : -t] \mid [s : t] \in \mathbb{P}^1\}.$$

If the family is independent of the parameter t, i.e., the equation defining the planes does not involve t, then we call this the *trivial* family. Meaning, all the planes in the family are a fixed projective plane P_0.

Problem 1.5: Consider the projective line

$$\ell := \{[s : t + s : t - s : t] \mid [t : s] \in \mathbb{P}^1_{\mathbb{R}}\}.$$

Find a family of projective planes P_t so that $\ell \subset P_t$ for every $t \in \mathbb{R}$, i.e., the line ℓ is a fixed line for the family of projective planes.

Can you find a family of projective lines in $\mathbb{P}^3_{\mathbb{R}}$ without a fixed point?

Can you find a family of projective planes in $\mathbb{P}^3_{\mathbb{R}}$ without a fixed point?
Solution 1.5:

A *projective plane* in $\mathbb{P}^3_{\mathbb{R}}$ is the space defined by a single equation of the form:

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$$

where not all the λ_i are zero.

Problem 1.5: Consider the projective plane H in $\mathbb{P}^3_{\mathbb{R}}$ defined by the equation

$$2x_0 + x_1 + x_2 + 2x_3 = 0.$$

Consider the projective line ℓ parametrized by

$$[s : s : t : t].$$

Find all the points of intersection of ℓ and H.

Solution 1.5:

In this problem, we will analyse Desargue's theorem in $\mathbb{R}^2$. The theorem is as follows.

Theorem (Desargues): Let two triangles ABC and $A'B'C'$ be such that the lines joining corresponding vertices, namely AA', BB', and CC', pass through a common point O. Then, the three pairs of corresponding sides intersect in three points

$$P = \overline{AB} \cap \overline{A'B'}$$
$$Q = \overline{BC} \cap \overline{B'C'}$$
$$R = \overline{AC} \cap \overline{A'C'}$$

which lie in a straight line. This means that the points P, Q, and R are collinear.

Problem 1.6: Draw two different situations in which the conditions of Desargues' theorem hold.

Check with your classmates that the statement of the theorem hold in this situation.

In case that anyone in the group knows a proof of Desargues theorem using Euclidean geometry. They must explain it to the rest of the students.

Solution 1.6:

Instead of considering two triangles $\Delta := ABC$ and $\Delta' := A'B'C'$ in $\mathbb{R}^2$ as in Desagues Theorem, we can consider this points in $\mathbb{P}^2_{\mathbb{R}}$. Or even in $\mathbb{P}^3_{\mathbb{R}}$. We will prove that Desargues' theorem holds for both spaces $\mathbb{P}^2_{\mathbb{R}}$ and $\mathbb{P}^3_{\mathbb{R}}$. When we prove Desargues' theorem in the projective plane (using projective geometry), we are also proving it in $\mathbb{R}^2$. Because three collinear points in $\mathbb{R}^2$ correspond to three collinear points in $\mathbb{P}^2_{\mathbb{R}}$. First, we will prove the theorem in the case that both triangles are in the projective space $\mathbb{P}^3_{\mathbb{R}}$. We will assume that the triangles are not contained in a common projective plane.

Desargues Theorem in the projective space: Consider the triangles Δ and Δ' in the projective space $\mathbb{P}^3_{\mathbb{R}}$. Each triangle Δ and Δ' are contained in two projective planes H and H'. Since we are assuming that the triangles are not contained in a commont projective plane, then $H \neq H'$. Two distinct planes in the projective space must intersect along a line. We write $H \cap H' = \ell$.

We argue that the lines $\overline{AB}$ and $\overline{A'B'}$ intersect at a point P. Indeed, both projective lines belong to the projective plane determined by the triangle OAB, so they must intersect at a point P. Analogously, we conclude that

$$\overline{AC} \cap \overline{A'C'} = R$$

and

$$\overline{BC} \cap \overline{B'C'} = Q.$$

Now, we argue that the points P, Q, and R are collinear. Note that $\overline{AB}$ is contained in H and $\overline{A'B'}$ is contained in H'. Then, the intersection of both lines must be contained in $H \cap H' = \ell$. Thus, P must be contained in ℓ. Analogously, we conclude that Q and R must be contained in the line ℓ. This finishes the proof: the three points P, Q, and R are contained in ℓ.

Problem 1.7: Draw the proof of Desargues' theorem for the projective space.
Solution 1.7:

Now, we turn to prove Desargues' theorem in the projective plane. To do so, we will use the case of Desargues' theorem in the projective space.

Desargues' Theorem in the projective plane: Consider the two triangles $\Delta := ABC$ and $\Delta' = A'B'C'$ in $\mathbb{P}^2_{\mathbb{R}}$. We can visualize the projective plane $H := \mathbb{P}^2_{\mathbb{R}}$ inside a projective space $\mathbb{P}^3_{\mathbb{R}}$. Consider a point X not contained in H. Draw the lines joining X to all the points in the diagram, i.e., O, A, B, C, A', B', and C'. Choose D in the projective line $\overline{XB}$, different from B. Let $D' = \overline{OD} \cap \overline{XB'}$. Then, the two triangles ADC and $A'D'C'$ are contained in different projective planes and are perspective from O. From the previous case, we conclude that the points

$$P' = \overline{AD} \cap \overline{A'D'}$$

$$Q = \overline{AC} \cap \overline{A'C'}$$

and

$$R' = \overline{DC} \cap \overline{D'C'}$$

are collinear. Then, there exists a unique plane H' containing P', Q, R', and X. The points P, Q, and R are contained in the line $H \cap H'$.

Problem 1.8: Draw the proof of Desargues' theorem in the projective plane.
Solution 1.8:

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