Spring Quarter - Worksheet 2: Projective Lines and Planes

Today we will study lines and planes in the projective space $\mathbb{P}^3_\mathbb{R}$. Often, we call $\mathbb{P}^1_\mathbb{R}$ the projective line, we call $\mathbb{P}^2_\mathbb{R}$ the projective plane, and we call $\mathbb{P}^3_\mathbb{R}$ the projective space. Thus, today we will be studying projective lines and planes in the projective space.

Problem 1.0: For the following triples of lines in $\mathbb{P}^2_\mathbb{R}$, draw how they intersect and at which points they intersect.

- $\ell_1 = \{x_0 = 0\}, \ell_2 = \{x_1 = 0\}$, and $\ell_3 = \{x_2 = 0\}$.
- $\ell_1 = \{x_0 + x_1 = 0\}, \ell_2 = \{x_0 + x_2 = 0\}$, and $\ell_3 = \{x_1 + x_2 = 0\}$.
- $\ell_1 = \{x_0 + x_1 + x_2 = 0\}, \ell_2 = \{x_0 + 2x_1 = 0\}$, and $\ell_3 = \{2x_0 + x_1 = 0\}$.

Solution 1.0:
Problem 1.1: Consider the line
\[ \ell := \{ [x_0 : x_1 : x_2] \mid x_0 + x_1 + x_2 = 0 \} \]
in the projective space \( \mathbb{P}^2_\mathbb{R} \). Find all the projective lines in \( \mathbb{P}^2_\mathbb{R} \) that intersect \( \ell \) only at infinity.

Solution 1.1:
Problem 1.2: Consider the point \( p := [0 : 1 : 0] \) in \( \mathbb{P}^2_{\mathbb{R}} \). Describe the set of all the projective lines \( \ell \) that pass through the point \( p \).

Solution 1.2:
A projective line in \( \mathbb{P}^3_{\mathbb{R}} \) is defined by two linear equations
\[
\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0 \quad \text{and} \quad \mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0,
\]
where \([\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3]\) and \([\mu_0 : \mu_1 : \mu_2 : \mu_3]\) are two different points in \( \mathbb{P}^3_{\mathbb{R}} \).

**Problem 1.3:** Describe the points in \( \mathbb{P}^3_{\mathbb{R}} \) satisfying the equations:
\[
x_0 + x_1 + x_2 + x_3 = 0,
\]
and
\[
x_0 - x_3 = 0.
\]

**Solution 1.3:**
Another way to define a projective line in \( \mathbb{P}^3_{\mathbb{R}} \) is \textit{parametrically}, i.e., via a function

\[
\mathbb{P}^1_{\mathbb{R}} \rightarrow \mathbb{P}^3_{\mathbb{R}}
\]

of the form:

\[
[s : t] \mapsto [l_0(s, t) : l_1(s, t) : l_2(s, t) : l_3(s, t)]
\]

where \( l_i \) are homogeneous linear functions in the variables \( s \) and \( t \).

**Problem 1.4:** Consider the projective line in \( \mathbb{P}^3_{\mathbb{R}} \) defined by the equations:

\[
x_0 + 2x_1 + x_2 = 0
\]

and

\[
x_1 + 2x_2 + x_3 = 0.
\]

Describe this projective line parametrically.

**Solution 1.4:**
A projective plane in \( \mathbb{P}^3_\mathbb{R} \) is the space defined by a single equation of the form:
\[
\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,
\]
where not all the \( \lambda_i \) are zero.

**Problem 1.5:** Consider the projective plane \( H \) in \( \mathbb{P}^3_\mathbb{R} \) defined by the equation
\[
2x_0 + x_1 + x_2 + 2x_3 = 0.
\]
Consider the projective line \( \ell \) parametrized by
\[
[s : s : t : t].
\]
Find all the points of intersection of \( \ell \) and \( H \).

**Solution 1.5:**
The plane given by \( \{ x_3 = 0 \} \) in \( \mathbb{P}_R^3 \) is called the \textit{hyperplane at infinity}.

**Problem 1.6:** Consider the hyperplane

\[ H := \{ x_0 + 2x_1 + 2x_2 + x_3 = 0 \} \]

in \( \mathbb{P}_R^3 \). Consider the projective line \( \ell \) described parametrically:

\[ [s : t : s : t]. \]

Describe the points at infinity of \( H \).
Describe the points at infinity of \( \ell \).
Describe the points of intersection of \( H \) and \( \ell \).

**Solution 1.6:**
We learnt that two projective lines always intersect in $\mathbb{P}^2_R$.

**Problem 1.7:** Find examples of two projective lines in $\mathbb{P}^3_R$ that do not intersect. Find examples of three projective lines in $\mathbb{P}^3_R$ that do not intersect.

**Solution 1.7:**
Problem 1.8: Show that two distinct planes in \( \mathbb{P}^3_{\mathbb{R}} \) always intersect along a projective line. Give an example in which this does not happen in \( \mathbb{R}^3 \). Give an example of two projective planes in the projective space \( \mathbb{P}^3_{\mathbb{R}} \) that only intersect at infinity. 

Solution 1.8:
Problem 1.9: Consider the planes

\[ H_0 := \{ x_0 = 0 \}, \quad H_1 := \{ x_1 = 0 \}, \quad H_2 := \{ x_2 = 0 \}, \quad \text{and} \quad H_3 := \{ x_3 = 0 \} \]

in \( \mathbb{P}^3 \). Find all the intersections \( H_i \cap H_j \) and triple intersections \( H_i \cap H_j \cap H_k \) of the previous projective planes. Is there any point in the projective space that lies in the intersection of all these projective planes? Give a geometric representation of how these planes intersect in the space.

Solution 1.9