OLGA RADKO MATH CIRCLE: ADVANCED 3

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Spring Quarter - Worksheet 2: Projective Lines and Planes

Today we will study lines and planes in the projective space $\mathbb{P}^3_{\mathbb{R}}$. Often, we call $\mathbb{P}^1_{\mathbb{R}}$ the *projective line*, we call $\mathbb{P}^2_{\mathbb{R}}$ the *projective plane*, and we call $\mathbb{P}^3_{\mathbb{R}}$ the *projective space*. Thus, today we will be studying projective lines and planes in the projective space.

Problem 1.0: For the following triples of lines in $\mathbb{P}^2_{\mathbb{R}}$, draw how they intersect and at which points they intersect.

- $\ell_1 = \{x_0 = 0\}, \ell_2 = \{x_1 = 0\}$, and $\ell_3 = \{x_2 = 0\}$.
- $\ell_1 = \{x_0 + x_1 = 0\}$, $\ell_2 = \{x_0 + x_2 = 0\}$, and $\ell_3 = \{x_1 + x_2 = 0\}$.
- $\ell_1 = \{x_0 + x_1 + x_2 = 0\}$, $\ell_2 = \{x_0 + 2x_1 = 0\}$, and $\ell_3 = \{2x_0 + x_1 = 0\}$.

Solution 1.0:

Problem 1.1: Consider the line

$$\ell := \{[x_0 : x_1 : x_2] \mid x_0 + x_1 + x_2 = 0\}$$

in the projective space $\mathbb{P}^2_{\mathbb{R}}$. Find all the projective lines in $\mathbb{P}^2_{\mathbb{R}}$ that intersect ℓ only at infinity.
Solution 1.1:

Problem 1.2: Consider the point $p := [0 : 1 : 0]$ in $\mathbb{P}^2_{\mathbb{R}}$. Describe the set of all the projective lines ℓ that pass through the point p.

Solution 1.2:

A *projective line* in $\mathbb{P}^3_{\mathbb{R}}$ is defined by two linear equations

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0 \qquad \text{and} \qquad \mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 = 0,$$

where $[\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3]$ and $[\mu_0 : \mu_1 : \mu_2 : \mu_3]$ are two different points in $\mathbb{P}^3_{\mathbb{R}}$.

Problem 1.3: Describe the points in $\mathbb{P}^3_{\mathbb{R}}$ satisfying the equations:

$$x_0 + x_1 + x_2 + x_3 = 0,$$

and

$$x_0 - x_3 = 0.$$

Solution 1.3:

Another way to define a projective line in $\mathbb{P}^3_{\mathbb{R}}$ is *parametrically*, i.e., via a function

$$\mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^3_{\mathbb{R}}$$

of the form:

$$[s:t] \mapsto [l_0(s,t) : l_1(s,t) : l_2(s,t) : l_3(s,t)]$$

where l_i are homogeneous linear functions in the variables s and t.

Problem 1.4: Consider the projective line in $\mathbb{P}^3_{\mathbb{R}}$ defined by the equations:

$$x_0 + 2x_1 + x_2 = 0$$

and

$$x_1 + 2x_2 + x_3 = 0.$$

Describe this projective line parametrically.
Solution 1.4:

A *projective plane* in $\mathbb{P}^3_{\mathbb{R}}$ is the space defined by a single equation of the form:

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$$

where not all the λ_i are zero.

Problem 1.5: Consider the projective plane H in $\mathbb{P}^3_{\mathbb{R}}$ defined by the equation

$$2x_0 + x_1 + x_2 + 2x_3 = 0.$$

Consider the projective line ℓ parametrized by

$$[s : s : t : t].$$

Find all the points of intersection of ℓ and H.
Solution 1.5:

The plane given by $\{x_3 = 0\}$ in $\mathbb{P}^3_{\mathbb{R}}$ is called the *hyperplane at infinity*.

Problem 1.6: Consider the hyperplane

$$H := \{x_0 + 2x_1 + 2x_2 + x_3 = 0\}$$

in $\mathbb{P}^3_{\mathbb{R}}$. Consider the projective line ℓ described parametrically:

$$[s : t : s : t].$$

Describe the points at infinity of H.
Describe the points at infinity of ℓ.
Describe the points of intersection of H and ℓ.
Solution 1.6:

We learnt that two projective lines always intersect in $\mathbb{P}^2_{\mathbb{R}}$.

Problem 1.7: Find examples of two projective lines in $\mathbb{P}^3_{\mathbb{R}}$ that do not intersect. Find examples of three projective lines in $\mathbb{P}^3_{\mathbb{R}}$ that do not intersect.

Solution 1.7:

Problem 1.8: Show that two distinct planes in $\mathbb{P}^3_{\mathbb{R}}$ always intersect along a projective line.
Give an example in which this does not happen in $\mathbb{R}^3$.
Give an example of two projective planes in the projective space $\mathbb{P}^3_{\mathbb{R}}$ that only intersect at infinity.
Solution 1.8:

Problem 1.9: Consider the planes

$$H_0 := \{x_0 = 0\}, \quad H_1 := \{x_1 = 0\}, \quad H_2 := \{x_2 = 0\}, \quad \text{and} \quad H_3 := \{x_3 = 0\}$$

in $\mathbb{P}^3_{\mathbb{R}}$. Find all the intersections $H_i \cap H_j$ and triple intersections $H_i \cap H_j \cap H_k$ of the previous projective planes. Is there any point in the projective space that lies in the intersection of all these projective planes?
Give a geometric representation of how these planes intersect in the space.

Solution 1.9

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