

# Lattices

Prepared by Mark on April 3, 2024

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**Definition 1:**

The *integer lattice*  $\mathbb{Z}^n \subset \mathbb{R}^n$  is the set of points with integer coordinates.

**Problem 2:**

Draw  $\mathbb{Z}^2$ .

**Definition 3:**

We say a set of vectors  $\{v_1, v_2, \dots, v_k\}$  *generates*  $\mathbb{Z}^n$  if every lattice point can be written uniquely as

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k$$

for integer coefficients  $a_i$ .

It is fairly easy to show that  $k$  must be at least  $n$ .

**Problem 4:**

Which of the following generate  $\mathbb{Z}^2$ ?

- $\{(1, 2), (2, 1)\}$
- $\{(1, 0), (0, 2)\}$
- $\{(1, 1), (1, 0), (0, 1)\}$

**Problem 5:**

Find a set of two vectors that generates  $\mathbb{Z}^2$ .

Don't say  $\{(0, 1), (1, 0)\}$ , that's too easy.

**Problem 6:**

Find a set of vectors that generates  $\mathbb{Z}^n$ .

**Definition 7:**

A *fundamental region* of a lattice is the parallelepiped spanned by a generating set. The exact shape of this region depends on the generating set we use.

**Problem 8:**

Draw two fundamental regions of  $\mathbb{Z}^2$  using two different generating sets. Verify that their volumes are the same.

## Part 1: Minkowski's Theorem

### Theorem 9: Blichfeldt's Theorem

Let  $X$  be a finite connected region. If the volume of  $X$  is greater than 1,  $X$  must contain two distinct points that differ by an element of  $\mathbb{Z}^n$ . In other words, there exist distinct  $x, y \in X$  so that  $x - y \in \mathbb{Z}^n$ .

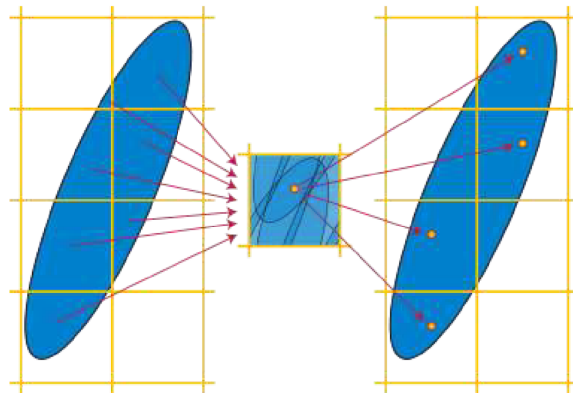
Intuitively, this means that you can translate  $X$  to cover two lattice points at the same time.

### Problem 10:

Draw a region in  $\mathbb{R}^2$  with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint:* Area is two-dimensional volume.

### Problem 11:

The following picture gives an idea for the proof of Blichfeldt's theorem in  $\mathbb{Z}^2$ . Explain the picture and complete the proof.



**Problem 12:**

Let  $X$  be a region  $\in \mathbb{R}^2$  of volume  $k$ . How many integral points must  $X$  contain after a translation?

**Definition 13:**

A region  $X$  is *convex* if the line segment connecting any two points in  $X$  lies entirely in  $X$ .

**Problem 14:**

- Draw a convex region in the plane.
- Draw a region that is not convex.

**Definition 15:**

We say a region  $X$  is *symmetric* if for all points  $x \in X$ ,  $-x$  is also in  $X$ .

**Problem 16:**

- Draw a symmetric region.
- Draw an asymmetric region.

**Theorem 17: Minkowski's Theorem**

Every convex set in  $\mathbb{R}^n$  that is symmetric with respect to the origin and which has a volume greater than  $2^n$  contains an integral point that isn't zero.

**Problem 18:**

Draw a few sets that satisfy Theorem 17 in  $\mathbb{R}^2$ .

What is the simplest region that has the properties listed above?

**Problem 19:**

Let  $K$  be a region in  $\mathbb{R}^2$  satisfying Theorem 17.

Let  $K'$  be this region scaled by  $\frac{1}{2}$ .

- How does the volume of  $K'$  compare to  $K$ ?
- Show that the sum of any two points in  $K'$  lies in  $K$  *Hint: Use convexity.*
- Apply Blichfeldt's theorem to  $K'$  to prove Minkowski's theorem in  $\mathbb{R}^2$ .

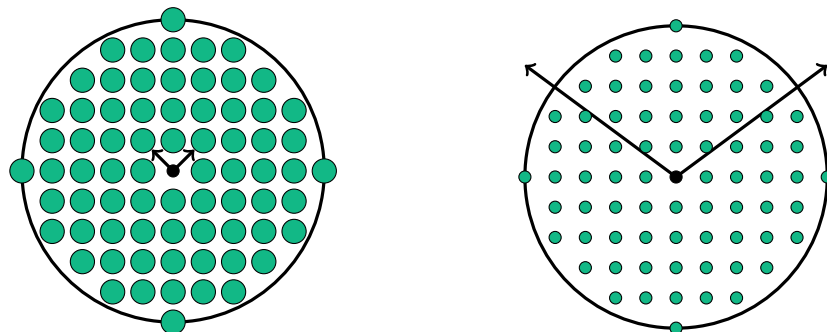
**Problem 20:**

Let  $K$  be a region in  $\mathbb{R}^n$  satisfying Theorem 17. Scale this region by  $\frac{1}{2}$ , called  $K' = \frac{1}{2}K$ .

- How does the volume of  $K'$  compare to  $K$ ?
- Show that the sum of any two points in  $K'$  lies in  $K$
- Apply Blichfeldt's theorem to  $K'$  to prove Minkowski's theorem.

## Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius  $R$ . A tree of radius  $r$  has been planted at every integer point in the circle. If  $r$  is small, you will have a clear line of sight through the orchard. If  $r$  is large, there will be no clear line of sight through in any direction:



### Problem 21:

Show that you will have at least one clear line of sight if  $r < \frac{1}{\sqrt{R^2+1}}$ .

*Hint:* Consider the line segment from  $(0, 0)$  to  $(R, 1)$ . Calculate the distance from the closest integer points to the ray.

**Problem 22:**

Show that there is no line of sight through the orchard if  $r > \frac{1}{R}$ . You may want to use the following steps:

- Show that there is no line of sight if  $r \geq 1$ .
- Suppose  $r < 1$  and  $r > \frac{1}{R}$ . Then,  $R \geq 2$ . Choose a potential line of sight passing through an arbitrary point  $P$  on the circle. Thicken this line of sight equally on both sides into a rectangle of width  $2r$  tangent to  $P$  and  $-P$ . From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

**Problem 23: Challenge**

Prove that there exists a rational approximation of  $\sqrt{3}$  within  $10^{-3}$  with denominator at most 501. Come up with an upper bound for the smallest denominator of a  $\epsilon$ -close rational approximation of any irrational number  $\alpha > 0$ . Your bound can have some dependence on  $\alpha$  and should get smaller as  $\alpha$  gets larger.

*Hint:* Use the orchard.