Lattices

Prepared by Mark on April 28, 2023

Definition 1:
The *integer lattice* $\mathbb{Z}^n \subset \mathbb{R}^n$ is the set of points with integer coordinates. We call each point in the lattice a *lattice point*.

Problem 2:
Draw $\mathbb{Z}^2$.

Definition 3:
We say a set of vectors $\{v_1, v_2, ..., v_n\}$ *generates* $\mathbb{Z}^n$ if every lattice point can be written uniquely as

$$a_1v_1 + a_2v_2 + ...a_nv_n$$

for integer coeficcients a_i.

Problem 4:
Which of the following generate $\mathbb{Z}^3$?
- $\{(1,2),(2,1)\}$
- $\{(1,0),(0,2)\}$
- $\{(1,1),(1,0),(0,1)\}$

Problem 5:
Find a set of vectors that generates $\mathbb{Z}^2$.

Problem 6:
Find a set of vectors that generates $\mathbb{Z}^n$.

Problem 7:
A *fundamental region* of a lattice is the parallelepiped spanned by a generating set. The exact shape of this region depends on the generating set we use.

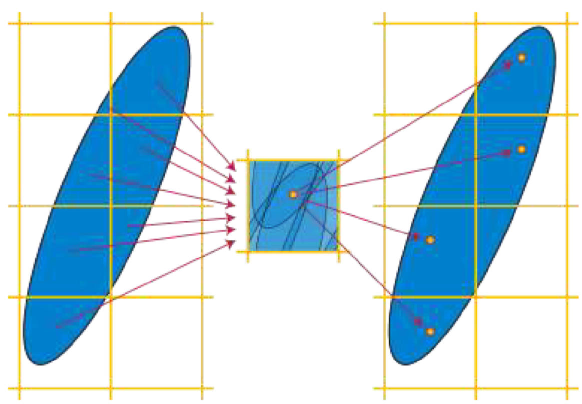
Problem 8:
Draw two fundamental reions of $\mathbb{Z}^2$ using two different generating sets. Verify that their volumes are the same.

Part 1: Minkowski's Theorem

Theorem 9: Blichfeldt's theorem
Let X be a finite connected region. If the volume of X is greater than 1, X must contain two distinct points that differ by an element of $\mathbb{Z}^n$. In other words, there exist distinct $x, y \in X$ so that $x - y \in \mathbb{Z}^n$.

Problem 10:
Draw a region in $\mathbb{R}^2$ with volume greater than 1 that contains no lattice points. Find two points in that region which differ by an integer vector. *Hint:* Area is two-dimensional volume.

Problem 11: Proof in $\mathbb{Z}^2$
The following picture gives the idea for the proof of Blichfeldt's theorem. Explain the picture and complete the proof.



Problem 12:
Let X be a region X of volume k. How many integral points must X contain after a translation?

Definition 13:
A region X is *convex* if the line segment connecting any two points in X lies entirely in X.

Problem 14:

- Draw a convex region in the plane.
- Draw a region that is not convex.

Definition 15:
We say a region is *symmetric with respect to the origin* if for all points x in the region, $-x$ is also in X.

Problem 16:

- Draw a symmetric region.
- Draw an asymmetric region.

Theorem 17: Minkowski's Theorem
Every convex set in $\mathbb{R}^n$ that is symmetric with respect to the origin and which has a volume greater than 2^n contains an integral point that isn't zero.

Problem 18:
Draw a few sets that satisfy Theorem 17 in $\mathbb{R}^2$. Which is the simplest region that has the properties listed above?

Problem 19:
Let K be a region in $\mathbb{R}^2$ satisfying Theorem 17. Scale this region by $\frac{1}{2}$, called $K' = \frac{1}{2}K$.

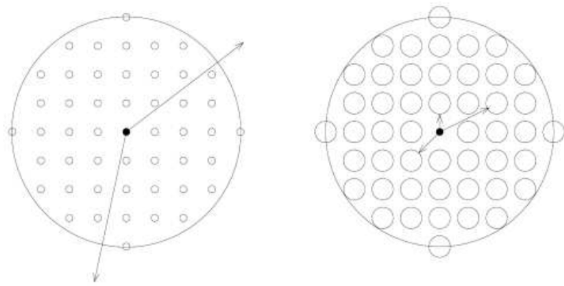
- How does the volume of K' compare to K?
- Show that the sum of any two points in K' lies in K
- Apply Blichfeldt's theorem to K' to prove Minkowski's theorem in $\mathbb{R}^2$.

Problem 20:
Let K be a region in $\mathbb{R}^n$ satisfying Theorem 17. Scale this region by $\frac{1}{2}$, called $K' = \frac{1}{2}K$.

- How does the volume of K' compare to K?
- Show that the sum of any two points in K' lies in K
- Apply Blichfeldt's theorem to K' to prove Minkowski's theorem.

Part 2: Polya's Orchard Problem

You are standing in the center of a circular orchard of integer radius R. A tree was planted each integer lattice point, and each has grown to the same radius r. If the radius is small enough, you will have a clear line of sight through the orchard in certain directions. If the radius is too large, there is no line of sight through the orchard in any direction. See the figure below:



Problem 21:
Show that if $r < \frac{1}{\sqrt{R^2+1}}$, you have at least one directon with a clear line of sight.
Hint: Take a look at the ray through the point $(R, 1)$ and calculate the distance from the closest integer points to the ray.

Problem 22:
Show that there is no line of sight through the orchard if $r > \frac{1}{R}$. You may want to use the following steps:

- Show that there is no line of sight if $r \geq 1$.
- Suppose $r < 1$ and $r > \frac{1}{R}$. Then, $R \geq 2$. Choose a potential line of sight passing through an arbitrary point P on the circle. Thicken this line of sight equally on both sides into a rectangle of width $2r$ tangent to P and $-P$. From here, use Minkowski's theorem to get a contradiction. Don't forget to rule out any lattice points that sit outside the orchard but inside the rectangle.

Problem 23: Challenge
Prove that there exists a rational approximation of $\sqrt{3}$ within 10^{-3} with denominator at most 501. Come up with an upper bound for the smallest denominator of a ϵ-close rational approximation of any irrational number $\alpha > 0$. Your bound can have some dependence on α and should get smaller as α gets larger.
Hint: Use the orchard.