# ORMC AMC 10/12 Training Week 5
# Number Theory II

April 30, 2023

## 1 Warm-Up

These problems should be a review of linear diophantine equations and the Chicken McNugget theorem along with other basic concepts.

1. Prove that if $\gcd(a_1, \ldots, a_n) = 1$, then every sufficiently large positive integer can be represented as a non-negative integer linear combination of the $a_i$.

2. Solve the Diophantine equation $x - y^4 = 4$, where x is a prime.

3. Find the largest integer $N$ such that no combination of coins worth $3, 10, 12$ is worth exactly $N$ units.

4. **(HMMT)** Compute the number of positive integers $n \leq 1000$ such that $\text{lcm}(n, 9)$ is a perfect square. (Recall that lcm denotes the least common multiple.)

5. **(Indian Math Olympiad)** For each positive integer $n$, let $s(n)$ denote the number of ordered pairs $(x, y)$ of positive integers for which
$$\frac{1}{x} + \frac{1}{y} = n.$$
Find all positive integers $n$ for which $s(n) = 5$.

6. **(Polish Math Olympiad)** Solve the following equation in integers x, y:
$$x^2(y-1) + y^2(x-1) = 1.$$

## 2 Continued Fractions

**Definition 1.** A (simple) continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where $a_i$ are non-negative integers for $i > 0$ and $a_0$ can be any integer. Since the above expression can be unwieldy, we can represent it in list notation as $[a_0, a_1, a_2, a_3, \dots]$. A finite continued fraction is one which has finitely many terms.

**Theorem 1.** Every rational number $\frac{a}{b}$ can be represented as a finite continued fraction.

*Proof.* Applying the Euclidean algorithm on $a$ and $b$ gives us the following system of equations:

$$\begin{aligned} a &= q_0 b + r_0 \\ b &= q_1 r_0 + r_1 \\ r_0 &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ r_{n-1} &= q_{n+1} r_n + 0. \end{aligned}$$

Let $[c_0, c_1, c_2, \dots]$ denote a continued fraction representation for $\frac{a}{b}$. Note that by definition of the division algorithm, $r_0 < b$ so $\left\lfloor \frac{a}{b} \right\rfloor = q_0 = c_0$. We are left with

$$\frac{a}{b} - c_0 = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \dots}}}$$

Thus,

$$\left(\frac{a}{b} - c_0\right)^{-1} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \dots}}$$

But

$$\frac{a}{b} - c_0 = \frac{a - bq_0}{b} = \frac{r_0}{b}.$$

Thus we have shown that $\frac{r_0}{b} = [c_1, c_2, \dots]$, and repeating the earlier argument shows $c_1 = q_1$. Using induction, it is clear that $c_i = q_i$ for all $i \geq 0$. Since the Euclidean algorithm terminates in a finite number of steps, it follows that every rational number has a finite continued fraction representation. $\square$

**Definition 2.** *Given a continued fraction* $[a_0, a_1, \dots]$*, the* $n$*-th convergent is defined to be* $\frac{h_n}{k_n} = [a_0, a_1, \dots, a_n]$*. For instance, in the above example, the first few convergents are:*

$$\begin{aligned} \frac{h_0}{k_0} &= c_0 \\ \frac{h_1}{k_1} &= c_0 + \frac{1}{c_1} \\ \frac{h_2}{k_2} &= c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2}} \end{aligned}$$

**Theorem 2.** Let $x_n = \frac{h_n}{k_n}$ denote the $n$-th convergent. Then the even convergents $x_{2n}$ increase strictly with $n$, while the odd convergents $x_{2n+1}$ decrease strictly.

*Proof.* We have that

$$\begin{aligned}
x_n - x_{n-2} &= \frac{h_n}{k_n} - \frac{h_{n-2}}{k_{n-2}} \\
&= \frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} + \frac{h_{n-1}}{k_{n-1}} - \frac{h_{n-2}}{k_{n-2}} \\
&= \frac{(-1)^{n-1}}{k_n k_{n-1}} + \frac{(-1)^{n-2}}{k_{n-1} k_{n-2}} \\
&= \frac{(-1)^n (k_n - k_{n-2})}{k_n k_{n-2} k_{n-1}} \\
&= \frac{(-1)^n a_n k_{n-1}}{k_n k_{n-2} k_{n-1}} \\
&= \frac{(-1)^n a_n}{k_n k_{n-2}}
\end{aligned}$$

This expression has the sign $(-1)^n$, which is positive when $n$ is even and negative when $n$ is odd. Thus, $x_{2n} < x_{2n+2}$ and $x_{2n-1} > x_{2n+1}$. □

## 2.1 Examples

1. Evaluate $[1, 1, 1]$ and $[1, 1, 1, \dots]$.

2. Find continued fractions for $\sqrt{2}$ and $\sqrt{3}$ to the third convergent.

3. Evaluate

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \dots}}}$$

## 2.2 Exercises

1. Find the periodic continued fraction for $\sqrt{a^2 + 1}$.

2. **(AHSME)** Solve $\sqrt{5 - x} = 5 - x^2$. (Hint: don't try to solve the quartic equation.)

3. Show that $|x - y\sqrt{d}| < 1$ is satisfied by an infinite number of pairs of positive integers (x,y).

4. Find the continued fractions for $\frac{100}{27}$ and $\frac{27}{100}$. In general, let $x = [a_0, \dots, a_n]$ and $1/x = [b_0, \dots, b_m]$. What can we conclude about the relationship between $a_i$ and $b_i$?

## 3 Pell's Equation

Last class, we dealt with a general class of equations called linear diophantine equations. Oftentimes, the equations we encounter are not so simple and may contain terms of higher degree. For instance, consider the quadratic diophantine equation $ax^2 + by^2 = c$.
The special case when $a = 1$, $b < 0$, $c = 1$ is known as **Pell's Equation**:

$$x^2 - ny^2 = 1.$$

**Theorem 3.** The equation $x^2 - ny^2 = 1$ has a non-trivial solution if and only if $n$ is not a square. If $n$ is not a square, then infinite solutions can be generated by the *fundamental solution* $(x_1, y_1)$, the solution with the smallest value of $x$ and $y$.

### 3.1 Proof (Exercises)

1. Show that if $n$ is a square then the equation $x^2 - ny^2 = 1$ has only the trivial solution $x = 1$ and $y = 0$.

2. Assume henceforth that $n$ is a non-square. Show that if $(x, y)$ is a solution to $x^2 - ny^2 = 1$, then the fraction $\frac{x}{y}$ is a close approximation for $\sqrt{n}$.

3. Let $\frac{h_i}{k_i}$ denote the $i$-th convergent of the continued fraction representation for $\sqrt{n}$. Then there exists an $i \in \mathbb{N}$ such that $x_1 = h_i$ and $y_1 = k_i$. That is, the fundamental solution can always be found by testing convergents until a solution is found. The proof for this statement is out of the scope for this lecture but will be assumed to be true. Using this method, find the fundamental solutions for $n = 2$ and $n = 3$.

4. Show that if $(x, y)$ and $(a, b)$ are two solutions to Pell's equation, then $(xa + nyb, xb + ya)$ is a third solution.

5. For all $k > 1$, define $(x_k, y_k) := x_k + y_k\sqrt{n} = (x_1 + y_1\sqrt{n})^k$. Show that $(x_k, y_k)$ is a solution to $x^2 - ny^2 = 1$. We say that $(x_k, y_k)$ is *generated* by $(x_1, y_1)$.

6. Show that all solutions are generated by the fundamental solution $(x_1, y_1)$.

## 3.2 Examples

1. Find the 3 smallest solutions to $x^2 - 2y^2 = 1$.

2. **(AMC 12)** A *triangular number* is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer $n$. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

## 3.3 Exercises

1. **(ARML)** Let $n$ be a positive integer, and consider the list $1, 2, 2, 3, 3, 3, ..., n, n, ..., n$ where the integer $k$ appears $k$ times in the list for $1 \leq k \leq n$. The integer $n$ will be called "ARMLy" if the median of the list is not an integer. The least ARMLy integer is 3. Compute the least ARMLy integer greater than 3.

2. Prove that if $n$ is a natural number and $(3n + 1)$ and $(4n + 1)$ are both perfect squares, then 56 will divide $n$.

3. **(AIME)** Find the largest integer $n$ satisfying the following conditions:
(i) $n^2$ can be expressed as the difference of two consecutive cubes;
(ii) $2n + 79$ is a perfect square.

4. **(British Math Olympiad)** Find the first integer $n > 1$ such that the average of $1^2, 2^2, \ldots, n^2$ is itself a perfect square.

5. **(European Girls Math Olympiad)** Let S be the set of all positive integers $n$ such that $n^4$ has a divisor in the range $n^2 + 1, n^2 + 2, \ldots, n^2 + 2n$. Prove that there are infinitely many elements of $S$ of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of $S$ of the form $7m + 3$ or $7m + 4$, where $m$ is an integer.