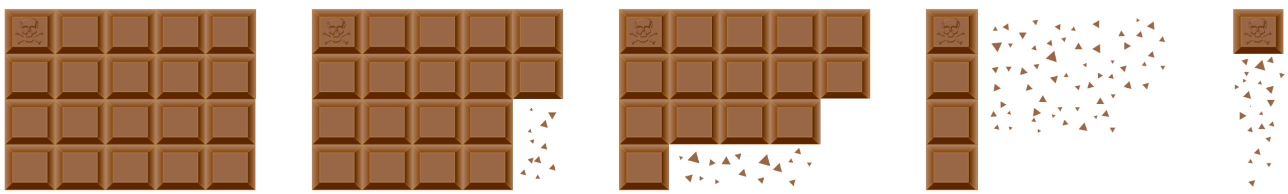
CHOMP

MAX STEINBERG FOR OLGA RADKO MATH CIRCLE
INTERMEDIATE 2

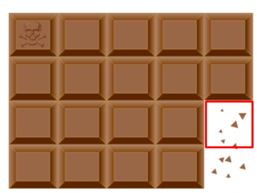
1. Chomp

Let's play a game. Consider a chocolate bar that is split up into squares. The top-left square is poisoned, so you don't want to eat it. I will go first. On my turn, I can select any square of the chocolate bar. I break off that square and any square below or to the right of it. Then you go and do the same. It is never allowed to choose to eat 0 squares. Whoever eats the poisoned square loses.



In the above picture, we start with a 5-by-4 chocolate bar. I started by eating two squares, and then you ate three squares. Then I ate eleven squares and you ate three, leaving me with the poisoned square. I am forced to eat it and you win the game!

Problem 1. In each of the pictures above, draw an outline around the specific square that each player selected on their turn. For example, on my first turn, I ate this square:



Problem 2. Play Chomp with the students sitting next to you. Try to find a strategy to win the game (a strategy may include selecting whether or not you wish to go first). Try playing on different grid sizes.

2. A Brief Interlude

Hopefully you remember how to win Nim and how to work with Nimbers. They will be very useful in our analysis of Chomp, so let's review. If you are unable to solve any of the following problems, that is totally alright! Just make sure to review the theory of Nim.

Problem 3. Who wins one-pile Nim?

Problem 4. Who wins in the 2-pile Nim game with stacks 2 and 3?

Problem 5. Who wins in the 3-pile Nim game with stacks 1, 2, and 3?

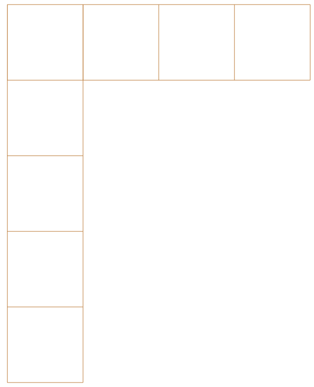
Problem 6. Calculate the Nim sum $1 \oplus 2 \oplus 4 \oplus 7$.

This problem will be very useful later, so make sure you understand the answer.

Problem 7. Consider the 2-pile Nim game with piles a and b, for some positive integers a and b. When is this game winning for Player 1, and when is this game winning for Player 2?

3. How to Win Chomp

Let's first consider a very simple example. We are playing Chomp and it is your turn. The current chocolate bar is L-shaped (shown below). The poisoned square, as always, is the top-left square, and there are 3 squares attached to the right and 4 squares attached below it.



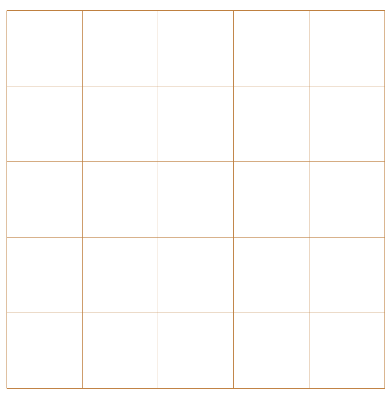
Problem 8. What are all the legal moves for you to make (ignoring taking the poison)?

Problem 9. Can you identify another game you know about with the same set of legal moves?

Problem 10. Who wins that game? Can you determine who would win our Chomp game from this position?

Problem 11. Given an L-shaped chocolate bar with a squares on the right and b squares below the poison, who wins?

Problem 12. Consider Chomp played on a *square* chocolate bar. Who wins the game? Why? (Feel free to play the game on the grid below to get a feel for the strategy.)
Hint: consider a **reduction argument**: try to reduce a complicated problem to a simpler case.



Let's try to win general (rectangular) Chomp. For notation, we write a-by-b Chomp to mean Chomp played on a chocolate bar of size a-by-b.

Problem 13. Who do you think wins a-by-b Chomp for a, b positive integers?

Problem 14. Let's play a-by-b Chomp. I will go first. Assume for the sake of argument that no matter what move I play, there is a move you can make that will make you win. Is this possible? Why or why not?
Hint: consider a **strategy-stealing argument**: if there was a move you could do to win, could I do that move instead? Would that make me win?
Conclude by determining who wins a-by-b Chomp.

4. Pmohc

Chomp got kinda boring, so I invented a brand-new game called Pmohc (pronounced "puh-mock")! Here's how we play: we start with a random positive integer that has exactly two unique prime factors (for example, we could start with $12 = 2^2 \cdot 3$ but not $60 = 2^2 \cdot 3 \cdot 5$). On your turn, you choose any divisor of the number given with two rules: you may not choose 1 and you may not choose an integer multiple that has already been chosen. If you have no moves, you lose. For example, say we start with 12. I start by selecting 2 (2 divides 12 so this is legitimate). Now, you can choose 3, since 3 divides 12 and 3 is not a multiple of 2. You may never pick 1, and you cannot pick 4, since 4 is a multiple of 2. So your only move is to pick 3. This leaves me with no moves and I lose.

Problem 15. Play Pmohc with the students next to you. Who do you think will win? (If you are having trouble finding numbers to start with, try $12, 72 = 2^3 \cdot 3^2, 2592 = 2^5 \cdot 3^4$. Do the choices of 2 and 3 as our primes matter? Do the exponents of 2 and 3 matter?)

Problem 16. Determine who wins Pmohc, given the starting number.

Hint: consider another reduction argument.

5. Difficult Problems

If you are finished doing all the above, but there still remains some time...

Problem 17. Who wins in three-dimensional Chomp? Four dimensions? n dimensions? What about Pmohc with an arbitrary integer?

Problem 18. Consider $\mathbb{N} = \{0, 1, \ldots, \}$, the natural numbers. Who wins Chomp played on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ (ie. ∞-by-∞-by-∞ Chomp)?

Next time, we will learn about another combinatorial game: Hackenbush!