Part 1: Notation and Terminology

Definition 1:
- \( \mathbb{R} \) is the set of all real numbers.
- \( \mathbb{R}^+ \) is the set of positive real numbers. Zero is not positive.
- \( \mathbb{R}_{\geq 0} \) is the set of positive real numbers and zero.

Mathematicians are often inconsistent with their notation. Depending on the author, their mood, and the phase of the moon, \( \mathbb{R}^+ \) may or may not include zero. We will use the definitions above.

Definition 2:
Consider two sets \( A \) and \( B \). The set \( A \times B \) consists of all tuples \( (a, b) \) where \( a \in A \) and \( b \in B \). For example, \( \{1, 2, 3\} \times \{\heartsuit, \star\} = \{(1, \heartsuit), (1, \star), (2, \heartsuit), (2, \star), (3, \heartsuit), (3, \star)\} \)

This is called the \textit{cartesian product}.

You can think of this as placing the two sets “perpendicular” to one another:

\[
\begin{array}{c}
\text{B} \\
\heartsuit & \star \\
1 & 2 & 3 & A \\
\end{array}
\]

Problem 1:
Let \( A = \{0, 1\} \times \{0, 1\} \)
Let \( B = \{a, b\} \)
What is \( A \times B \)?

Problem 2:
What is \( \mathbb{R} \times \mathbb{R} \)?
\textit{Hint:} Use the “perpendicular” analogy
Definition 3:
$\mathbb{R}^n$ is the set of $n$-tuples of real numbers.
In English, this means that an element of $\mathbb{R}^n$ is a list of $n$ real numbers:

Elements of $\mathbb{R}^2$ look like $(a, b)$, where $a, b \in \mathbb{R}$.
Elements of $\mathbb{R}^5$ look like $(a_1, a_2, a_3, a_4, a_5)$, where $a_n \in \mathbb{R}$.
$\mathbb{R}^1$ and $\mathbb{R}$ are identical.

Intuitively, $\mathbb{R}^2$ forms a two-dimensional plane, and $\mathbb{R}^3$ forms a three-dimensional space.
$\mathbb{R}^n$ is hard to visualize when $n \geq 4$, but you are welcome to try.

Problem 3:
Convince yourself that $\mathbb{R} \times \mathbb{R}$ is $\mathbb{R}^2$.
What is $\mathbb{R}^2 \times \mathbb{R}$?
Part 2: Vectors

Definition 4:
Elements of \( \mathbb{R}^n \) are often called vectors.

As you may already know, we have a few operations on vectors:

- Vector addition: \([a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]\)
- Scalar multiplication: \(x \times [a_1, a_2] = [xa_1, xa_2]\).

The above examples are for \( \mathbb{R}^2 \), and each vector thus has two components. These operations are similar for all other \( n \).

Problem 4:
Compute the following or explain why you can’t:

- \([1, 2, 3] - [1, 3, 4]\) Subtraction works just like addition.
- \(4 \times [5, 2, 4]\)
- \(a + b\), where \( a \in \mathbb{R}^5 \) and \( b \in \mathbb{R}^7\)

Problem 5:
Consider \((2, -1)\) and \((3, 1)\) in \( \mathbb{R}^2 \).
Can you develop geometric intuition for their sum and difference?
**Definition 5: Euclidean Norm**

A norm on \(\mathbb{R}^n\) is a map from \(\mathbb{R}^n\) to \(\mathbb{R}_0^+\).

Usually, one thinks of a norm as a way of measuring “length” in a vector space.

The norm of a vector \(v\) is written \(||v||\).

We usually use the Euclidean norm when we work in \(\mathbb{R}^n\).

If \(v \in \mathbb{R}^n\), the Euclidean norm is defined as follows:

If \(v = [v_1, v_2, ..., v_n]\),

\[ ||v|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2} \]

This is simply an application of the Pythagorean theorem.

**Problem 6:**

Compute the euclidean norm of

- \([2, 3]\)
- \([-2, 1, -4, 2]\)

**Problem 7:**

Show that \(a \cdot a\) is \(||a||^2\).
Part 3: Dot Products

Definition 6:
We can also define the dot product of two vectors.\footnote{Bonus content. Feel free to skip.}
The dot product maps two elements of $\mathbb{R}^n$ to one element of $\mathbb{R}$:
\[
    a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + ... + a_n b_n
\]

Problem 8:
Compute $[2, 3, 4, 1] \cdot [2, 4, 10, 12]$

Problem 9:
Show that the dot product is
- Commutative
- Distributive $a \cdot (b + c) = a \cdot b + a \cdot c$
- Homogenous: $x(a \cdot b) = xa \cdot b = a \cdot xb$
  \[x \in \mathbb{R}, \text{ and } a, b \text{ are vectors.}\]
- Positive definite: $a \cdot a \geq 0$, with equality iff $a = 0$
  \[a \in \mathbb{R}^n, \text{ and } 0 \text{ is the zero vector.}\]
Problem 10:
Say you have two vectors, $a$ and $b$. Show that $a \cdot b = ||a|| \ ||b|| \cos(\alpha)$, where $\alpha$ is the angle between $a$ and $b$.

*Hint:* What is $c$ in terms of $a$ and $b$?

*Hint:* The law of cosines is $a^2 + b^2 - 2ab \cos(\alpha) = c^2$

*Hint:* The length of $a$ is $||a||$

Problem 11:
If $a$ and $b$ are perpendicular, what must $a \cdot b$ be? Is the converse true?
Part 4: Matrices

Definition 7:
A \textit{matrix} is a two-dimensional array of numbers:

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

The above matrix has two rows and three columns. It is thus a $2 \times 3$ matrix.
The order “first rows, then columns” is usually consistent in linear algebra.
If you look closely, you may also find it in the next definition.

Definition 8:
We can define the product of a matrix $A$ and a vector $v$:

\[
Av = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix}
1a + 2b + 3c \\
4a + 5b + 6c
\end{bmatrix}
\]

Note that each element of the resulting $2 \times 1$ matrix is the dot product of a row of $A$ with $v$:

\[
Av = \begin{bmatrix}
-r_1 \\
r_2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix}
r_1 \cdot v \\
r_2 \cdot v
\end{bmatrix}
\]

Naturally, a vector can only be multiplied by a matrix if the number of rows in the vector equals the number of columns in the matrix.

Problem 12:
Compute the following:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\begin{bmatrix}
5 \\
3
\end{bmatrix}
\]

Problem 13:
Say you multiply a size-$m$ vector $v$ by an $m \times n$ matrix $A$.
What is the size of your result $Av$?
Definition 9:
We can also multiply a matrix by a matrix:

\[
AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 100 & 200 \end{bmatrix} = \begin{bmatrix} 210 & 420 \\ 430 & 860 \end{bmatrix}
\]

Note each element of the resulting matrix is dot product of a row of \( A \) and a column of \( B \):

\[
AB = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 \cdot v_1 & r_1 \cdot v_2 \\ r_2 \cdot v_1 & r_2 \cdot v_2 \end{bmatrix}
\]

Problem 14:
Compute the following matrix product.

\[
\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}
\]

Problem 15:
Compute the following matrix product or explain why you can’t.

\[
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}
\]

Problem 16:
If \( A \) is an \( m \times n \) matrix and \( B \) is a \( p \times q \) matrix, when does the product \( AB \) exist?
Problem 17:
Is matrix multiplication commutative?
Does $AB = BA$ for all $A, B$?
You only need one counterexample to show this is false.

Definition 10:
Say we have a matrix $A$. The matrix $A^T$, pronounced “A-transpose”, is created by turning rows of $A$ into columns, and columns into rows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Problem 18:
Compute the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} 1 \\ 3 \\ 3 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix}^T$$
The “transpose” operator is often used to write column vectors in a compact way. Vertical arrays don’t look good in horizontal text.

**Problem 19:**
Consider the vectors $a = [1, 4, 3]^T$ and $b = [9, 1, 4]^T$
- Compute the dot product $a \cdot b$.
- Can you redefine the dot product using matrix multiplication?

As you may have noticed, a vector is a special case of a matrix.

**Problem 20:**
A *column vector* is an $m \times 1$ matrix.
A *row vector* is a $1 \times m$ matrix.
We usually use column vectors. Why?
*Hint:* How does vector-matrix multiplication work?
Part 5: Bonus

**Problem 21:**
Show that the euclidean norm satisfies the triangle inequality:

\[ \|x + y\| \leq \|x\| + \|y\| \]

**Problem 22:**
Show that the euclidean norm satisfies the reverse triangle inequality:

\[ \|x - y\| \geq \|\|x\| - \|y\|\| \]

**Problem 23:**
Prove the Cauchy-Schwartz inequality:

\[ \|x \cdot y\| \leq \|x\| \|y\| \]