Combinatorics I

Welcome to Math Circle (Intermediate) in...how many ways?

September 30, 2012

1. Suppose there are 5 different teacups, 3 different tea saucers, and 4 different teaspoons in the "Tea Party" store. How many ways are there to buy a set consisting of a cup, a saucer, and a spoon?

2. Suppose we revist the "Tea Party" store from problem 1. How many ways are there to buy two different items from the set? For example, we could buy one of the cups and one of the spoons, but not two spoons.

3. There are four towns A, B, C, and D in Wonderland. In Wonderland, all roads are one-way. Suppose there are 6 different roads from A to B, 3 from A to C, 4 B to C, and 2 C to D. How many ways are there to get from A to C?

4. Each box in a 8×8 checkerboard can be colored either black or red.

   (a) Assuming that two colorings that can be obtained from each other by rotating the board are counted as different, how many different colorings of the checkerboard are there?
   (Note that we are *not* requiring the board to be exactly half black and half red.)

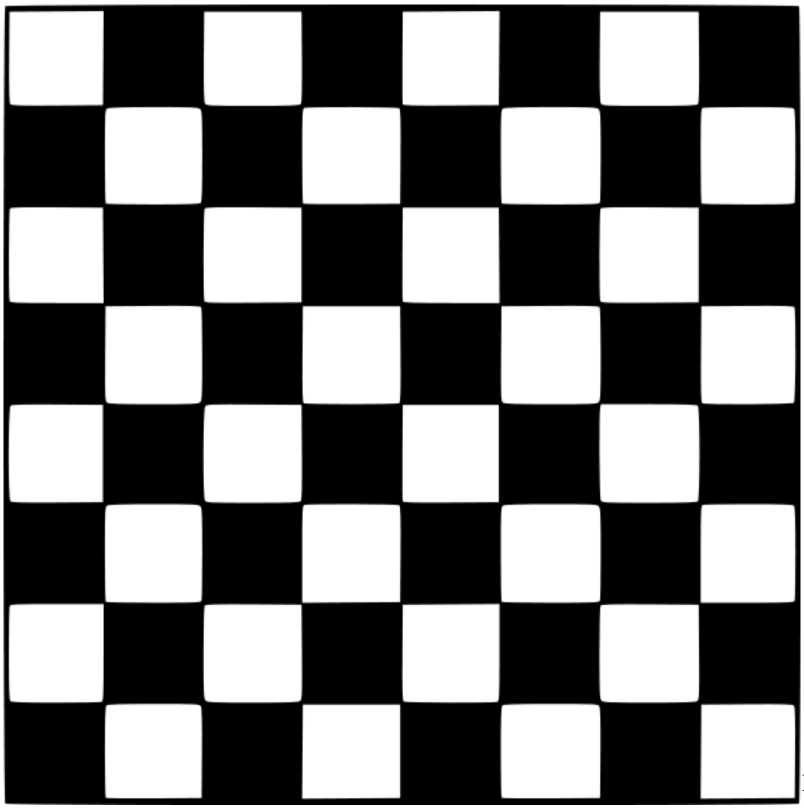
   (b) How many colorings do you have if you count two colorings that can be obtained from each other by rotating the board as the same?

5. Suppose we have a ternary alphabet with only the letters A, B, and C. A word in this language is an arbitrary sequence of no more than 4 letters. How many words does the language contain?

6. A captain and a deputy captain must be elected in a soccer team with 11 players. How many ways are there to do this?

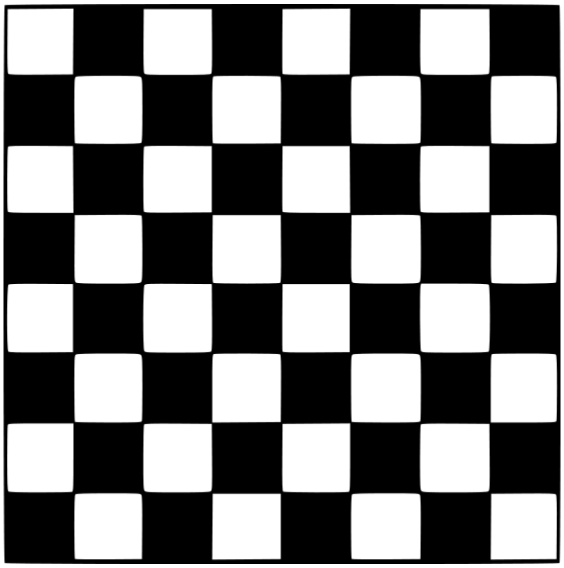
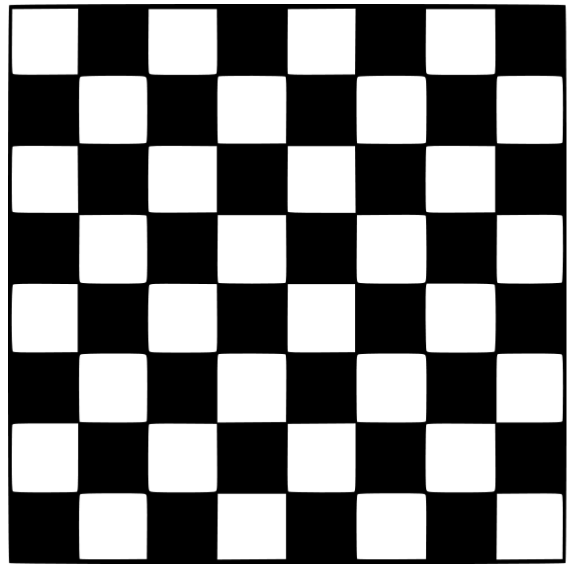
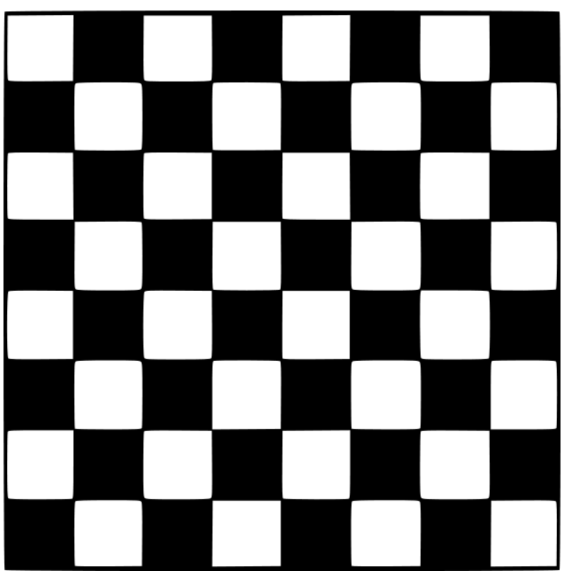
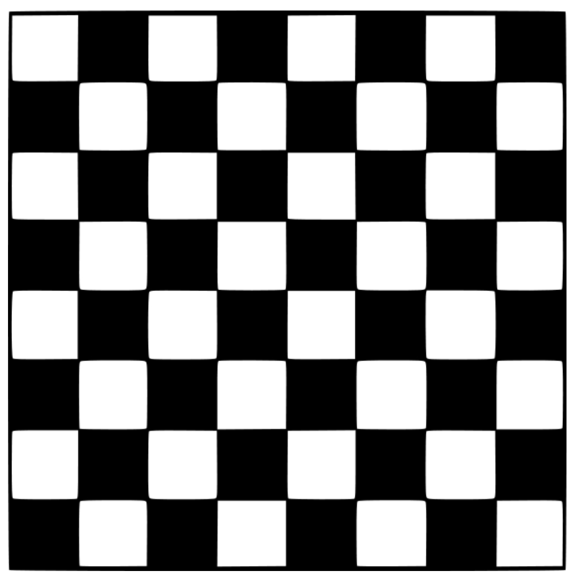
7. Suppose we have pieces of fabric of 6 colors. How many ways are there to sew a flag with 3 horizontal strips (each of a different color) of equal height? Note that we can distinguish the top of the flag from the bottom.

8. How many ways are there to put 2 rooks on a chessboard so that they do not attack each other?

[1]

[1]Source: math.cornell.edu

9. How many ways are there to put 2 bishops on a chessboard so that they do not attack each other?
Here are a few chessboards that you can draw on for help...

If n is a natural number, then n! (said n factorial) is the product

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

For example, 2! = 2; 3! = 6, and 4! = 24. We also define 0! = 1.

10. Simplify the following expressions:

$$10! \cdot 11 \quad =$$

$$n! \cdot (n+1) \quad =$$

11. Calculate $\frac{100!}{98!}$. Then compute $\frac{n!}{(n-1)!}$ for any natural number n.

12. Prove that if p is a prime number, then $(p-1)!$ is not divisible by p.

13. How many ways are there to lay four balls, colored red, blue, yellow, and green, in a row?

14. How many different "words" can be obtained by rearranging the letters of the word "VECTOR"?

15. (a) How many different "words" can be obtained using rearranging the letters of the word "TRUST"? (*Hint:* The problem is different from the previous one. Can you see why?)

    (b) How about the word "CARAVAN"?

Generally, if we have n_1 letters of A_1, n_2 letters of A_2, . . . , and n_k letters of A_k, there are

$$\frac{(n_1 + ... + n_k)!}{n_1!...n_k!}$$

different words formed using all of the letters.

16. (a) How many different words can be obtained using rearranging the letters of "CLOSENESS"? Express your answer in terms of factorials.

    (b) How about the longest word to ever appear in an English dictionary, "PNEUMONOULTRAMICROSCOPICSILICOVOLCANOCONIOSIS"? Express your answer in terms of factorials.

17. (a) There are 4 towns in a certain country, and every pair of them is connected by an air route. How many air routes are there?
    Do your counting in TWO different ways and explain why they give the same answer.

    (b) There are 20 towns in another country, and every pair of them is connected by an air route. How many air routes are there in this country?

(c) What if there were n towns in the country?

18. How many diagonals are there in an n-gon (a polygon of n sides)? (A diagonal is a segment connecting two vertices of the polygon which are not next to each other. Note that for some n-gons, some of the diagonals may lie outside of the polygon.)

19. How many ways are there to seat 13 people at a round table? (Assume that the table cannot be rotated.)

20. How many five-digit numbers have at least one even digit? (Hint: You may find it easier to first calculate how many five-digit numbers have *no* even digits.)

[2]

[2]Some problems are taken from:
D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"