Graph Theory Continued

Last time, we learned about (simple) graphs and how they can represent a wide range of phenomena in the world. Studying the mathematical properties of graphs in turn lets us better understand corresponding real-world problems. The properties we learned about last time are: vertex degree, completeness, whether there are various types of paths, and connectivity. But are these properties enough to understand everything we want to know about graphs?

Graph isomorphism

The main question we want to think about is: when do two graphs represent the same physical objects and their connections? It is not clear at first whether this is an easy or hard question to answer. Let's start with an example. Consider the three graphs below: A, B, and C.

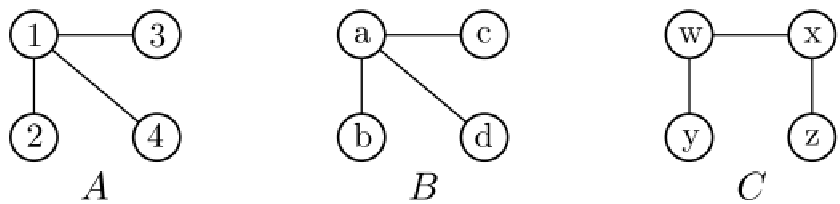


Figure 1: Graphs A, B, and C

Graphs A and B clearly visually represent the same phenomena. Indeed, we can match vertex 1 with vertex a, vertex 2 with vertex b, vertex 3 with vertex c, and vertex 4 with vertex d. Moreover, for each edge in graph A, there is a corresponding edge between the matching vertices in graph B. The same is true vice versa! We can think of graphs A and B as being the "same", just labeled differently.

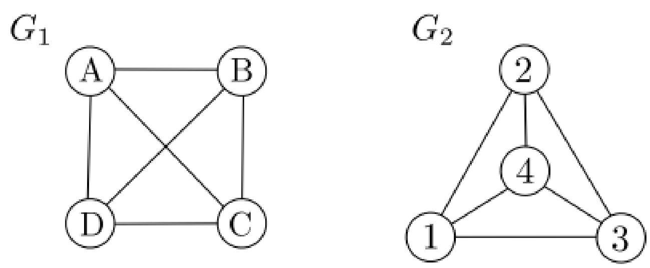
But what about graph C? Notice that vertex 1 from graph A is of degree 3, meaning it has three edges coming out of it. But every vertex in graph C only is of degree 1 or 2; there are no vertices with three edges coming out. So then if we try to match vertex 1 with any vertex in graph C, we won't be able to match the edges of the graphs together. This means that graph A and C cannot be the "same", and hence do not represent the same physical connections between real-world objects.

This example motivates us to give the following definition. Two graphs (without loops and multiple edges) are called **isomorphic**, if it is possible to enumerate the vertices of each graph with the same set of numbers so that the following condition is fulfilled: if two vertices in the first graph are connected by an edge, then the vertices in the second graph with the same numbers are connected by an edge, and vice versa.

Another way of saying this is that two graphs A and B are isomorphic if there exists a matching between their vertices so that two vertices are connected by an edge in A only when the matching two vertices in B are also connected by an edge, and vice versa.

Overall, the word isomorphic is just a fancy mathematical way of saying that two graphs represent the same objects and connections. Now, it was easy to tell which graphs in Figure 1 are the same, and we even smartly used a property we learned about last time: vertex degree. However, let's try thinking about some hard examples.

Problem 1. *The graphs G_1 and G_2 below are isomorphic.*

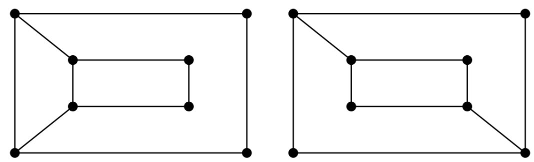


Complete the following parts.

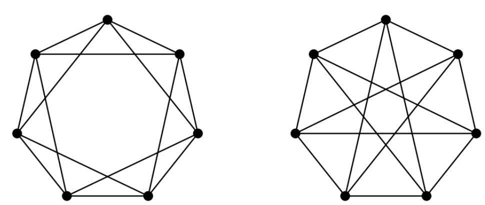
(i) *Write down which vertices in G_1 match with which vertices in G_2.*

(ii) *Label the edges in G_1 and the ones in G_2. Then, write down which edges in G_1 match with which edges in G_2.*

Problem 2. *Label the edges and vertices on the graphs below. Then say whether or not these two graphs are isomorphic.*



Problem 3. *Label the edges and vertices on the graphs below. Then say whether or not these two graphs are isomorphic.*



Now you can see it is getting more difficult to tell when graphs are isomorphic. Let's try to see when certain graph properties are helpful in identifying isomorphic graphs.

Problem 4. *Suppose you have two graphs A and B that are isomorphic. Answer yes or no to each of the following parts. If your answer is no, give an example. Note you do not need to provide rigorous proofs.*

(i) *Do A and B have to have the same number of vertices?*

(ii) *Do A and B have to have the same number of edges?*

(iii) *Do matching vertices in A and B have to have the same degree?*

(iv) *If A is connected, does B also have to be connected?*

(v) *If A has a cycle, does the matching path in B also have to be a cycle?*

(vi) *If A has an Eulerian cycle, does B also have to have an Eulerian cycle?*

Problem 5. *Suppose you have two graphs A and B. Answer yes or no to each of the following parts. If your answer is no, give an example. Note you do not need to provide rigorous proofs.*

(i) *If A and B have the same number of vertices, do they have to be isomorphic?*

(ii) *If A and B have the same number of edges, do they have to be isomorphic?*

(iii) *Suppose A and B have the same number of degree 0 vertices, the same number of degree 1 vertices, the same number of degree 2 vertices, and so on. Do A and B have to be isomorphic?*

This problem continues to the next page.

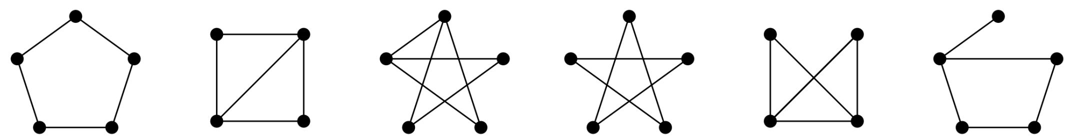
This is a continuation of the problem on the prior page.

(iv) *Suppose A and B have the same number of vertices and edges. If A and B are also both connected, do A and B have to be isomorphic?*

(vi) *Suppose A and B have the same number of vertices and edges. If A and B also both have Eulerian cycles, do A and B have to be isomorphic?*

Problem 4 shows that there are many "sufficient conditions" for graphs to be isomorphic. However, Problem 5 in turn shows that just because two graphs satisfy those conditions, it doesn't mean the two graphs have to be isomorphic. These two problems together show that it is usually very difficult to answer whether or not two graphs are isomorphic, especially if they are huge graphs. Nonetheless, we can still work with very small graphs. Use what you've learned to answer the following two problems.

Problem 6. *Let A be the left-most graph below, let B be the second graph, let C be the third graph, and so on until the right-most graph F. Divide the graphs into groups based on if they are isomorphic or not. To get you started, A and B are not in the same group because they are not isomorphic.*



Problem 7. *Draw all mutually non-isomorphic graphs with at most four vertices.*