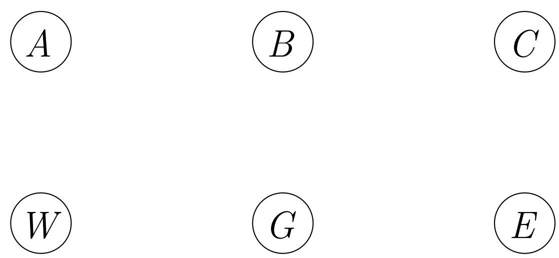
Planar Graphs

Problem 1. *There are three houses A, B, and C. Each house needs water from the facility W, gas from the facility G, and electricity from the facility E. Can you draw lines connecting each house to each of the facilities below so that the lines do not intersect?*



Last time we learned about **graph isomorphism**, which is when two drawings of a graph represent the same underlying objects and connections. To jog our memory, the graphs G_1 and G_2 are **isomorphic** (meaning the same):

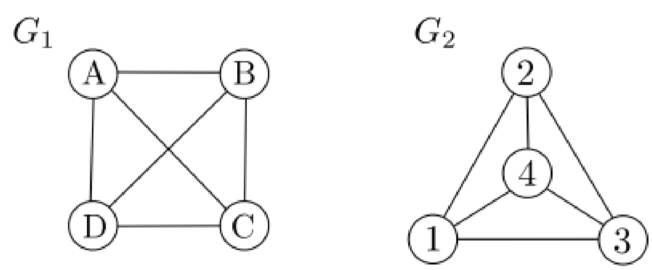


Figure 1: Isomorphic graphs

Indeed, we can match vertices A and 1 together, B and 2 together, C and 3 together, and D and 4 together so that the corresponding edges match too. Even though these graphs are the same, their drawings look quite different. This makes us ask, what properties of graphs do not change no matter how we draw them? Alternatively, what can we change about the graph by drawing it in a different (but isomorphic) way?

Crossings and Crossing Number

Remember that a graph has a **vertex set** of objects and an **edge set** of connections between the objects. When we draw a graph, we draw the vertices as points and the edges as lines connecting the points. Of course, the lines representing edges will intersect at the points representing vertices. However, sometimes the lines also intersect at points that do not represent vertices, and we call these extra intersections **crossings**. For example, let's look at Figure 1. The vertices in G_1 are A, B, C, D. But, the way we drew G_1 has a crossing in the middle, since the edge connecting A to C and the edge connecting B to D intersect at a point that is not A, B, C, or D.

Drawings of graphs with lots of crossings are harder to work with. So, our goal will be to understand if crossings have to exist and if there is any way to remove them.

Problem 2. *Complete the following parts about the graphs in Figure 1.*

(i) *How many crossings does the drawing G_1 have? How many crossings does the drawing G_2 have?*

(ii) *Draw a graph below that is also isomorphic to G_1 and G_2, but has two crossings. If you don't think it is possible, say so below.*

(iii) *Draw a graph below that is also isomorphic to G_1 and G_2, but has three crossings. If you don't think it is possible, say so below.*

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(iv) *Draw a graph below that is also isomorphic to G_1 and G_2, but has five crossings. If you don't think it is possible, say so below.*

(v) *Draw a graph below that is also isomorphic to G_1 and G_2, but has ten crossings. If you don't think it is possible, say so below.*

(vi) *Is there a maximum to the number of crossings a graph can have? Why or why not?*

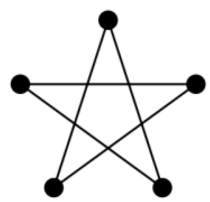
Problem 2 teaches us that two drawings of the same graph can have different numbers of crossings. We can also make the drawing of a graph as complicated as we want by adding crossings. But can we simplify a drawing by removing all crossings?

The **crossing number** of a graph G is the smallest number of crossings that a drawing of G must always have. In Figure 1 we see that we can remove all of the crossings from the drawing, and so G_1 and G_2 (which are the same graph) have crossing number equal to 0.

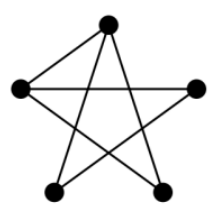
If every graph has crossing number equal to 0, that means we can draw any graph as simply as we'd like. However, if there is a graph with crossing number equal to 1, then the best we can do is draw that graph with 1 crossing. In general, if there is a graph with crossing number equal to $n > 0$, then the best we can do is draw that graph with n crossings. So, let's try to draw some graphs to figure out their crossing numbers.

Problem 3. *Draw each graph below in a way that you think minimizes the number of crossings. Then, write down what you think the crossing number is. Feel free to draw many versions of each graph until you are confident you cannot remove any more crossings. Compare with your neighbors.*

(i)



(ii)



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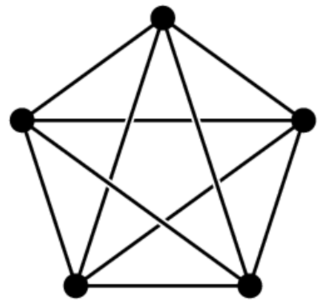
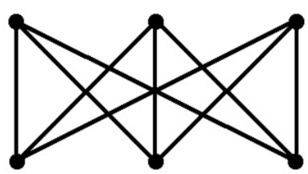
(iii)



(iv)



Planar graphs and Euler characteristic

After Problem 3, we should be reasonably sure that some graphs do not have crossing number equal to 0! This makes the graphs that do have crossing number equal to 0 special, and so we give them the special name of **planar graphs**. We use the word planar because we can draw the graphs on the flat 2-dimensional plane without any crossings.

Something we can define with planar graphs but not with non-planar graphs is the notion of a graph **face**. Indeed, let G be a connected planar graph, drawn with no crossings. The edges of G divide the plane into regions called faces. There are two types of faces of G: the one surrounding the graph is called the **exterior (or infinite) face** and the rest, which are all enclosed by the graph, are called the **interior faces**. For example, the following graph has two interior faces, F_1, bounded by the edges e_1, e_2, e_4; and F_2, bounded by the edges e_1, e_3, e_4. Its exterior face, F_3, is bounded by the edges e_2, e_3.

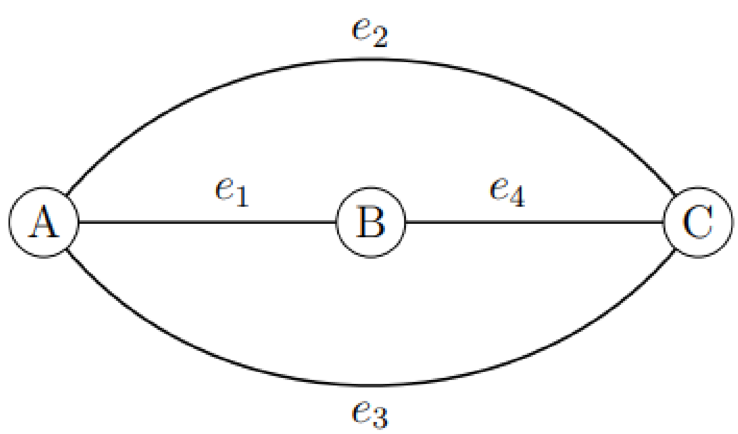


Figure 2: Example of a planar graph

In turn, we can let V be the number of vertices in the planar graph, E be the number of edges, and F be the number of faces. The **Euler characteristic** (written χ) of a graph is the number of vertices minus the number of edges plus the number of faces:

$$\chi = V - E + F.$$

Note that, no matter how you draw a planar graph, the number of faces won't change as long as there are no crossings in the drawing.

Problem 4. *Complete the following parts about the graph in Figure 2.*

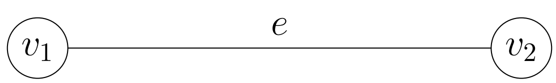
(i) *What is the number of vertices?* $V =$ ____________________

(ii) *What is the number of edges?* $E =$ ____________________

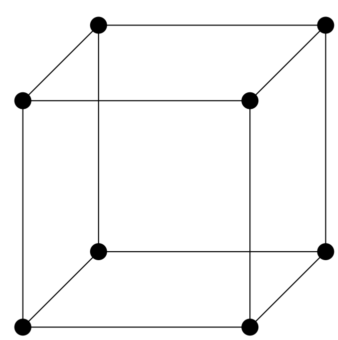
(iii) *What is the number of faces?* $F =$ ____________________

(iv) *What is the Euler characteristic?* $\chi =$ ____________________

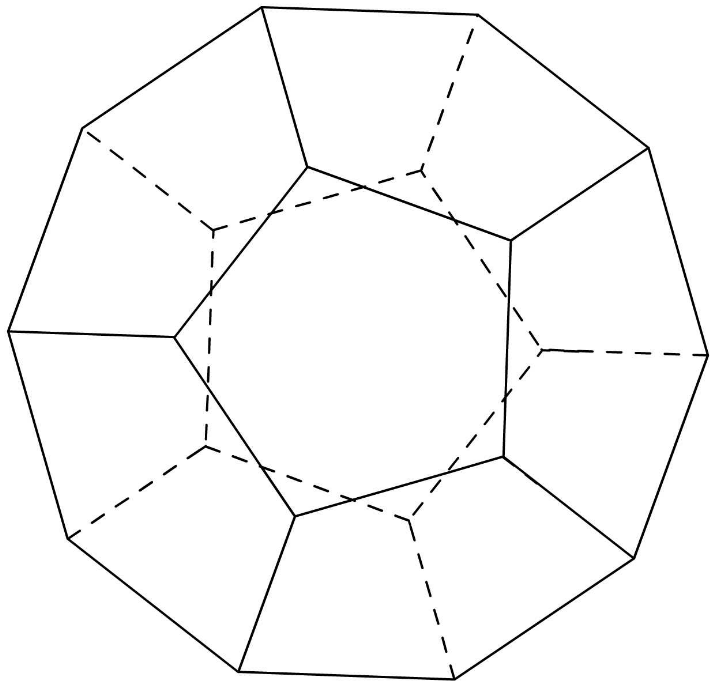
Problem 5. *Compute the Euler characteristic of the following graph.*



Problem 6. *Is the following graph planar? If you think it is, please re-draw the graph so that it has no crossings and compute the Euler characteristic. If you think the graph is not planar, please explain why.*



Problem 7. *Consider the regular dodecahedron as a graph with the vertices of the polytope representing those of the graph, the edges of the polytope (both solid and dashed) representing the edges of the graph.*



Is the following graph planar? If you think it is, please re-draw the graph so that it has no crossings and compute the Euler characteristic. If you think the graph is not planar, please explain why.

After these many examples, we are led to believe the following theorem.

Theorem 1. *If G is a connected planar graph, then the Euler characteristic $\chi = V - E + F$ has to equal the number $\chi = \underline{\qquad\qquad}$.*

Problem 8. *Let G be a connected planar graph with V_G many vertices, E_G many edges, and F_G many faces. We walk through a (semi-rigorous) inductive proof of Theorem 1. Let us induct on E, the number of edges.*

Base case:

We consider the base case where $E = 0$, meaning the graph G has zero edges. Then, G has to be the graph with only one vertex and no edges. So, we have

$$V = 1, \qquad E = 0, \qquad F = \underline{\qquad}.$$

This lets us compute the Euler characteristic to be

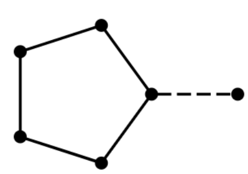
$$\chi = V - E + F = 1 - 0 + \underline{\qquad} = \underline{\qquad}.$$

Inductive Step:

Let $n \geq 1$ be a random integer. Assume we have a graph H with n many edges such that Theorem 1 works for H. This means that $V_H - E_H + F_H = \underline{\qquad}$ where V_H is the number of vertices in H, $E_H = n$ is the number of edges in H, and F_H is the number of faces in H. Next, suppose we add an edge to H creating a new graph G that is still planar. Then, G is still connected and has $n + 1$ edges. We want to show that Theorem 1 also works for G. That is, we want to show that $V_G - E_G + F_G = \underline{\qquad}$ where V_G is the number of vertices in G, $E_G = n + 1$ is the number of edges in G, and F_G is the number of faces in G. There are two situations to consider.

Situation 1:

In this situation, the edge that we added is connected to only one vertex in H. So, we had to also add a vertex along with the edge, like in the image below:



In this example, H is the pentagon with $n = 5$ edges. We added the dashed edge and vertex to create G, without making any new faces. So,

$$V_G = V_H + \_\_\_\_\_, \qquad E_G = n + 1 = E_H + 1, \qquad F_G = F_H + \_\_\_\_\_.$$

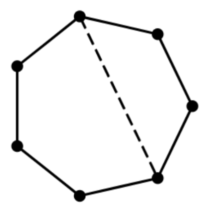
Then,

$$\begin{aligned} V_G - E_G + F_G &= V_H + \_\_\_\_\_ - (E_H + 1) + F_H + \_\_\_\_\_ \\ &= V_H - E_H + F_H \\ &= \_\_\_\_\_, \end{aligned}$$

because we know that Theorem 1 works for H. This shows that Theorem 1 also works for G in this first situation.

Situation 2:

In this situation, the edge we added connected two vertices already in H. So, we cut one face of H into two faces of G, like in the image below:



In this example, H is the heptagon with $n = 7$ edges. We added the dashed edge to create G, without adding any vertices but instead splitting one face of H into two faces of G. So,

$$V_G = V_H + \_\_\_\_\_, \qquad E_G = n + 1 = E_H + 1, \qquad F_G = F_H + \_\_\_\_\_.$$

Then,

$$\begin{aligned} V_G - E_G + F_G &= V_H + \_\_\_\_\_ - (E_H + 1) + F_H + \_\_\_\_\_ \\ &= V_H - E_H + F_H \\ &= \_\_\_\_\_, \end{aligned}$$

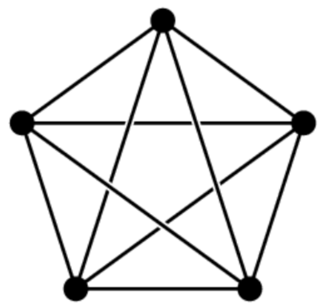
because we know that Theorem 1 works for H. This shows that Theorem 1 also works for G in this second situation.

Conclusion:

We've finished the base case and inductive step, so we know that Theorem 1 is proven true by induction.

Theorem 1 is a very important theorem since it lets us prove that some graphs are not planar!

Problem 9. *Complete the following parts about the graph drawn below, which is called K_5.*



(i) *What is the number of vertices?* $V =$ ______.

(ii) *What is the number of edges?* $E =$ ______.

(iii) *Now, let's pretend that we know K_5 is planar. Then, by Theorem 1, What is the number of faces?*

$$F = \chi - V + E = \underline{\hspace{3cm}}$$

(iii) *Let's keep pretending that K_5 is planar. Then, each face of K_5 has to have at least three edges as its boundary. Each edge also lies between two faces. These two facts mean that the number of edges E is at least 3 times the number of faces F divided by 2. So, substituting our answer to **(iii)**, we get*

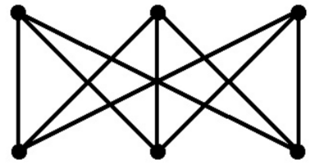
$$E \geq \frac{3F}{2} = \frac{3 \times \underline{\hspace{1cm}}}{2} = \underline{\hspace{1cm}}.$$

(iv) *So, by our answers to **(ii)** and **(iii)**, we know that if K_5 is planar, then*

$$\underline{\hspace{1cm}} = E \geq \underline{\hspace{1cm}},$$

which is impossible! This means that K_5 cannot be planar (since K_5 being planar implies something impossible). So, the crossing number of K_5 is at least 1. Go back to Problem 3 part (iii) and try to draw the graph with exactly one crossing.

Problem 10. *Complete the following parts about the graph drawn below.*



This graph is called $K_{3,3}$ because it is made by taking two rows of 3 vertices and then connecting each of the vertices in the top row to the vertices in the bottom row, and vice versa. Notice that none of the vertices in the top row are connected to each other, and similarly for the vertices in the bottom row.

(i) *What is the number of vertices?* $V =$ ______.

(ii) *What is the number of edges?* $E =$ ______.

(iii) *Now, let's pretend that we know $K_{3,3}$ is planar. Then, by Theorem 1, What is the number of faces?*

$$F = \chi - V + E = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(iii) *Let's keep pretending that $K_{3,3}$ is planar. Then just as for K_5, each face of $K_{3,3}$ has to have at least three edges as its boundary. For $K_{3,3}$, however, we can exploit the fact that vertices in the top and bottom rows don't connect to other vertices in the same row. This means each face of $K_{3,3}$ has to have at least four edges as its boundary. Examine the drawing above to verify this.*

*This fact, along with every edge lying between two faces, means that the number of edges E is at least 4 times the number of faces F divided by 2. So, substituting our answer to **(iii)**, we get*

$$E \geq \frac{4F}{2} = 2 \times \_\_\_\_\_\_ = \_\_\_\_\_\_.$$

(iv) *So, by our answers to **(ii)** and **(iii)**, we know that if $K_{3,3}$ is planar, then*

$$\_\_\_\_\_\_ = E \geq \_\_\_\_\_\_,$$

which is impossible! This means that $K_{3,3}$ cannot be planar (since $K_{3,3}$ being planar implies something impossible). So, the crossing number of $K_{3,3}$ is at least 1. Go back to Problem 3 part (iv) and try to draw the graph with exactly one crossing. Lastly, circle the answer to Problem 1: YES / NO.