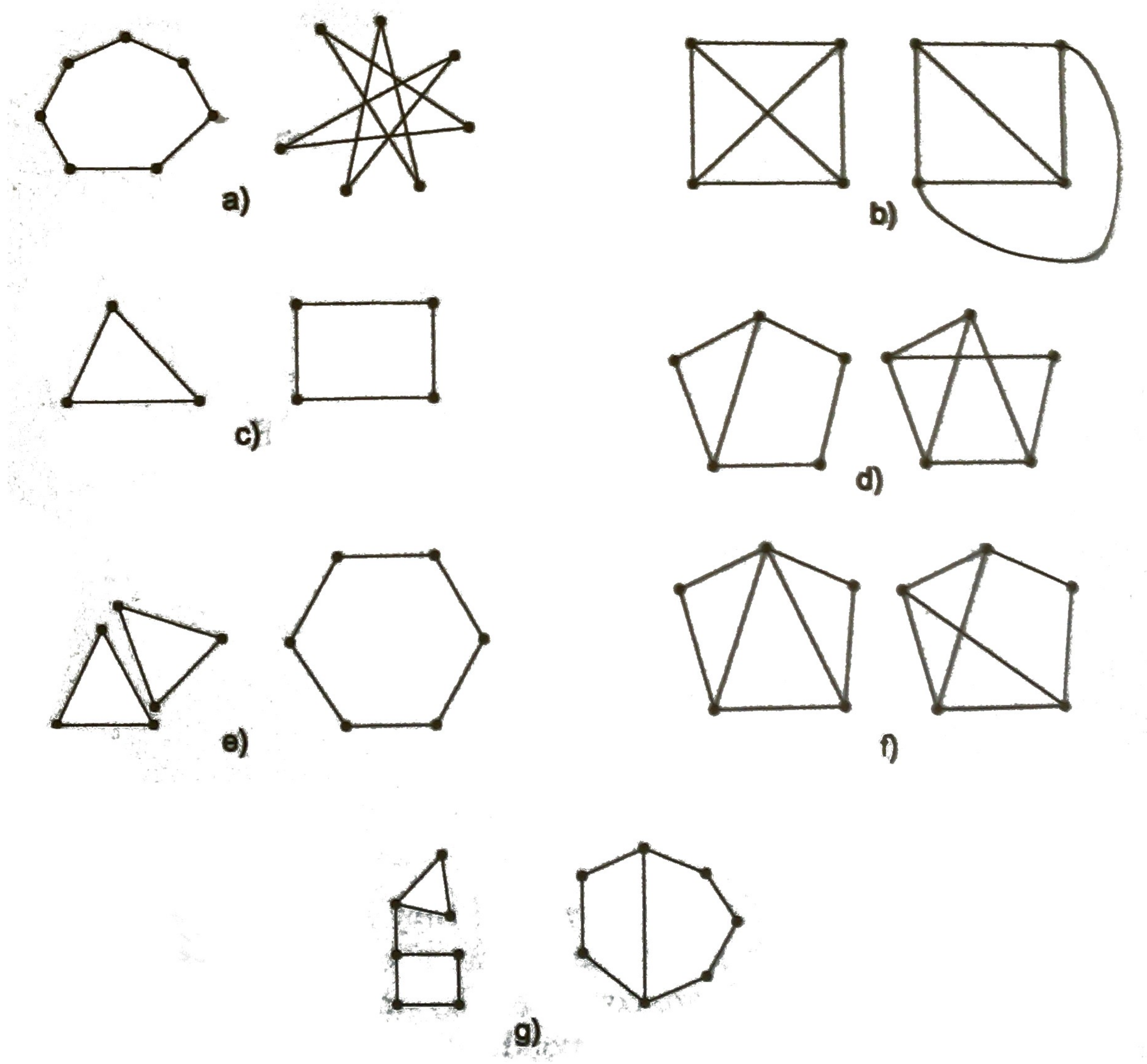
GRAPHS, ISOMORPHISM, AND PATHS

MATH CIRCLE (ADVANCED) 9/25/2012

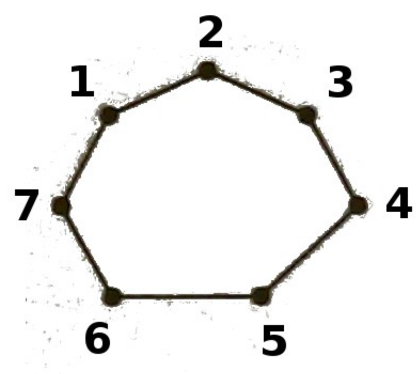
Recall that in mathematics, a graph is made up of a set of objects, where some of the objects are connected by links. The objects are called the *vertices* of the graph, and the links between the vertices are called *edges*.

0) Below is a collection of graphs. How many vertices and edges do each of the graphs have?



1) Do you think we should consider any of the preceding graphs the *same*? Why or why not?

Hint: For example, we could think of the following graph



as $G = (V, E)$, where $V = \{1, 2, \ldots, 7\}$, and $E = \{(1, 2), (2, 3), \ldots, (6, 7), (1, 7)\}$. Note that the way we label the vertices does NOT matter.

2) Is it true that two graphs must be isomorphic, if

a) they both have 10 vertices and the degree of each equals 9?

b) they are both connected, without cycles, and have 6 edges?

c)* they both have 8 vertices and the degree of each equals 3?

If we can draw a graph (i.e. draw all the edges) without picking up our pencil, we say that the graph has an *Eulerian path*. If the path starts and ends in the same place, we call it an *Eulerian cycle*.

3)

a) Which of the graphs on the first page have Eulerian paths? cycles?

b) Can you come up with a rule for deciding whether an arbitrary graph has Eulerian paths and/or cycles? What does this say about our warmup problem?

Hint: Think about the degrees of the vertices in the graph.

c)* Prove it!

4) Recall the setup for the warmup problem. Suppose the blue prince, red prince, and bishop can each build one road (in that order). Where should each build their road?

a) The blue prince wants to be able to walk all (eight) bridges, starting at his castle and ending at the inn.

b) The red prince wants to be able to walk all (nine) bridges, starting at his castle and ending at the inn. He also wants to make it impossible for the blue prince to do the same.

c) The bishop wants everyone to be able to walk all (ten) bridges, starting at their own home and ending at the inn.

5)

a) Suppose we have 120 cm of wire. Prove that it is not possible to use it to form a cube, each with side lengths 10 cm, without cutting the wire.

b) How many cuts are needed to actually form the cube?

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
- For more information on the setup from the warmup problem, see the wikipedia page: http://en.wikipedia.org/wiki/Seven_Bridges_of_Konigsberg