ORMC AMC 10/12 Training Week 3
Graph Theory

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Definition 1. The *complete graph* on n vertices, denoted by K_n, is the graph with n vertices such that there is an edge connecting every pair of distinct vertices.

Definition 2. A graph is *bipartite* if its vertices V can be partitioned into two disjoint subsets V_1 and V_2 such that no edge has both endpoints within the same set. A graph is *complete bipartite* if it is bipartite and all possible edges between V_1 and V_2 are drawn. The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

Definition 3. A graph is *planar* if it can be drawn on a plane such that no two edges intersect (except at the vertices). Convince yourself that Euler's formula $V + F - E = 2$ for polyhedrons also holds for planar graphs.

Theorem 1 (Handshaking Lemma)**.** Let $G = (V, E)$ be a graph. Then

$$2|E| = \sum_{v \in V} d(v).$$

Theorem 2 (Hall's Marriage Theorem)**.** Let $G = (X, Y, E)$ be a finite bipartite graph. For each $W \subset X$ define the *neighborhood* of W as

$$N_G(W) = \{v \in Y : \text{there is } w \in W \text{ such that } v \text{ and } w \text{ are adjacent}\}.$$

Suppose for any $w \in W$ there is $|W| \leq |N_G(W)|$. Then G has a perfect matching, i.e., a set of disjoint edges that matches every vertex in X with every vertex in Y. To be more precise, a *perfect matching* is a bijective function $\varphi : X \to Y$ such that x and $\varphi(x)$ is connected by an edge in G for any $x \in X$.

Theorem 3 (Euler's Formula)**.** Let $G = (V, E)$ be a planar graph. Then

$$V + F - E = 2.$$

In particular, F is the number of faces, where a *face* is any region bounded by edges, including the outer, infinitely large region.

(You do not have to prove the theorems.)

Exercise 1 (2012 AIME II #14)**.** In a group of nine people each person shakes hands with exactly two of the other people from the group. Let N be the number of ways this handshaking can occur. Consider two handshaking arrangements different if and only if at least two people who shake hands under one arrangement do not shake hands under the other arrangement. Find the remainder when N is divided by 1000.

Exercise 2. Consider an 8×8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

Exercise 3 (China 2016)**.** There are 10 points in $\mathbb{R}^3$ such that no three are on the same line and no four are on the same plane. What is the maximum number of edges that can be drawn among these points such that they do not form triangles or quadrilaterals?

Exercise 4. Let G be a connected graph with an even number of vertices. Prove that you can select a subset of edges of G such that each vertex is incident to an odd number of the selected edges.

Exercise 5 (USAMO 2007)**.** Consider the grid of unit squares in the plane. A cell is any unit square in the grid. An animal with n cells is a connected figure consisting of n cells (two cells are connected if and only if they share a common side). A k-saur is an animal with at least k cells. It is said to be primitive if its cells cannot be partitioned into two or more k-saurs. Find with proof the maximum number of cells in a primitive k-saur.

Exercise 6 (Turán Theorem)**.** A simple graph G has n vertices. Prove that if G does not contain a triangle, then G has at most $\lfloor n^2/4 \rfloor$ edges.

Exercise 7 (Italy 2007)**.** Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to color the computers and cables such that no two computers have the same color, no two cables joined to a common computer have the same color, and no computer is assigned the same color as any cable joined to it. Prove that this can be done using n colors.

Exercise 8. 20 football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

Exercise 9. Let n be a positive integer satisfying the following property: If n dominoes are placed on a 6 $\times$ 6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n.

Exercise 10 (USAMO 2023 Problem 3)**.** Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2 - 1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find all possible values of $k(C)$ as a function of n.